

Manual on Presentation of Data and Control Chart Analysis

8th Edition

Dean V. Neubauer, Editor

ASTM E11.90.03 Publications Chair

ASTM Stock Number: MNL7-8TH

Prepared by

Committee E11 on Quality and Statistics

Revision of Special Technical Publication (STP) 15D



INTERNATIONAL

Standards Worldwide

ASTM International
100 Barr Harbor Drive
PO Box C700
West Conshohocken, PA 19428-2959

Printed in U.S.A.

Library of Congress Cataloging-in-Publication Data

Manual on presentation of data and control chart analysis / prepared by Committee E11 on Quality and Statistics. — 8th ed.
p. cm.

Includes bibliographical references and index.

“Revision of special technical publication (STP) 15D.”

ISBN 978-0-8031-7016-2

1. Materials—Testing—Handbooks, manuals, etc. 2. Quality control—Statistical methods—Handbooks, manuals, etc.

I. ASTM Committee E11 on Quality and Statistics. II. Series.

TA410.M355 2010

620.1'10287—dc22

2010027227

Copyright © 2010 ASTM International, West Conshohocken, PA. All rights reserved. This material may not be reproduced or copied, in whole or in part, in any printed, mechanical, electronic, film, or other distribution and storage media, without the written consent of the publisher.

Photocopy Rights

Authorization to photocopy items for internal, personal, or educational classroom use of specific clients is granted by ASTM International provided that the appropriate fee is paid to ASTM International, 100 Barr Harbor Drive, PO Box C700, West Conshohocken, PA 19428-2959, Tel: 610-832-9634; online: <http://www.astm.org/copyright/>

ASTM International is not responsible, as a body, for the statements and opinions advanced in the publication. ASTM does not endorse any products represented in this publication.

Printed in Newburyport, MA
August, 2010

Preface

This *Manual on the Presentation of Data and Control Chart Analysis* (MNL 7) was prepared by ASTM's Committee E11 on Quality and Statistics to make available to the ASTM membership, and others, information regarding statistical and quality control methods and to make recommendations for their application in the engineering work of the Society. The quality control methods considered herein are those methods that have been developed on a statistical basis to control the quality of product through the proper relation of specification, production, and inspection as parts of a continuing process.

The purposes for which the Society was founded—the promotion of knowledge of the materials of engineering and the standardization of specifications and the methods of testing—involve at every turn the collection, analysis, interpretation, and presentation of quantitative data. Such data form an important part of the source material used in arriving at new knowledge and in selecting standards of quality and methods of testing that are adequate, satisfactory, and economic, from the standpoints of the producer and the consumer.

Broadly, the three general objects of gathering engineering data are to discover: (1) physical constants and frequency distributions, (2) the relationships—both functional and statistical—between two or more variables, and (3) causes of observed phenomena. Under these general headings, the following more specific objectives in the work of ASTM may be cited: (a) to discover the distributions of quality characteristics of materials that serve as a basis for setting economic standards of quality, for comparing the relative merits of two or more materials for a particular use, for controlling quality at desired levels, and for predicting what variations in quality may be expected in subsequently produced material, and to discover the distributions of the errors of measurement for particular test methods, which serve as a basis for comparing the relative merits of two or more methods of testing, for specifying the precision and accuracy of standard tests, and for setting up economical testing and sampling procedures; (b) to discover the relationship between two or more properties of a material, such as density and tensile strength; and (c) to discover physical causes of the behavior of materials under particular service conditions, to discover the causes of nonconformance with specified standards in order to make possible the elimination of assignable causes and the attainment of economic control of quality.

Problems falling in these categories can be treated advantageously by the application of statistical methods and quality control methods. This Manual limits itself to several of the items mentioned under (a). **PART 1** discusses frequency distributions, simple statistical measures, and the presentation, in concise form, of the essential information contained in a single set of n observations. **PART 2** discusses the problem of expressing plus and minus limits of uncertainty for various statistical measures, together with some working rules for rounding-off observed results to an appropriate number of significant figures. **PART 3** discusses the control chart method for the analysis of observational data obtained from a series of samples and for detecting lack of statistical control of quality.

The present Manual is the eighth edition of earlier work on the subject. The original *ASTM Manual on Presentation of Data*, *STP 15*, issued in 1933, was prepared by a special committee of former Subcommittee IX on Interpretation and Presentation of Data of ASTM Committee E01 on Methods of Testing. In 1935, Supplement A on Presenting Plus and Minus Limits of Uncertainty of an Observed Average and Supplement B on "Control Chart" Method of Analysis and Presentation of Data were issued. These were combined with the original manual, and the whole, with minor modifications, was issued as a single volume in 1937. The personnel of the Manual Committee that undertook this early work were H. F. Dodge, W. C. Chancellor, J. T. McKenzie, R. F. Passano, H. G. Romig, R. T. Webster, and A. E. R. Westman. They were aided in their work by the ready cooperation of the Joint Committee on the Development of Applications of Statistics in Engineering and Manufacturing (sponsored by ASTM International and the American Society of Mechanical Engineers [ASME]) and especially of the chairman of the Joint Committee, W. A. Shewhart. The nomenclature and symbolism used in this early work were adopted in 1941 and 1942 in the American War Standards on Quality Control (Z1.1, Z1.2, and Z1.3) of the American Standards Association, and its Supplement B was reproduced as an appendix with one of these standards.

In 1946, ASTM Technical Committee E11 on Quality Control of Materials was established under the chairmanship of H. F. Dodge, and the Manual became its responsibility. A major revision was issued in 1951 as *ASTM Manual on Quality Control of Materials*, *STP 15C*. The Task Group that undertook the revision of **PART 1** consisted of R. F. Passano, Chairman, H. F. Dodge, A. C. Holman, and J. T. McKenzie. The same task group also revised **PART 2** (the old Supplement A) and the task group for revision of **PART 3** (the old Supplement B) consisted of A. E. R. Westman, Chairman, H. F. Dodge, A. I. Peterson, H. G. Romig, and L. E. Simon. In this 1951 revision, the term "confidence limits" was introduced and constants for computing 95 % confidence limits were added to the constants for 90 % and 99 % confidence limits presented in prior printings. Separate treatment was given to control charts for "number of defectives," "number of defects," and "number of defects per unit," and material on control charts for individuals was added. In subsequent editions, the term "defective" has been replaced by "nonconforming unit" and "defect" by "nonconformity" to agree with definitions adopted by the American Society for Quality Control in 1978. (See the American National Standard, ANSI/ASQC A1-1987, *Definitions, Symbols, Formulas and Tables for Control Charts*.)

There were more printings of *ASTM STP 15C*, one in 1956 and a second in 1960. The first added the ASTM Recommended Practice for Choice of Sample Size to Estimate the Average Quality of a Lot or Process (E122) as an Appendix. This recommended practice had been prepared by a task group of ASTM Committee E11 consisting of A. G. Scroggie, Chairman, C. A. Bicking, W. E. Deming, H. F. Dodge, and S. B. Littauer. This Appendix was removed from that edition because it is revised more often than the main text of this Manual. The current version of E122, as well as of other relevant ASTM publications, may be procured from ASTM. (See the list of references at the back of this Manual.)

In the 1960 printing, a number of minor modifications were made by an ad hoc committee consisting of Harold Dodge, Chairman, Simon Collier, R. H. Ede, R. J. Hader, and E. G. Olds.

The principal change in *ASTM STP 15C* introduced in *ASTM STP 15D* was the redefinition of the sample standard deviation to be $s = \sqrt{\sum (X_i - \bar{X})^2 / (n-1)}$. This change required numerous changes throughout the Manual in mathematical equations and formulas, tables, and numerical illustrations. It also led to a sharpening of distinctions between sample values, universe values, and standard values that were not formerly deemed necessary.

New material added in *ASTM STP 15D* included the following items. The sample measure of kurtosis, g_2 , was introduced. This addition led to a revision of Table 1.8 and Section 1.34 of **PART 1**. In **PART 2**, a brief discussion of the determination of confidence limits for a universe standard deviation and a universe proportion was included. The Task Group responsible for this fourth revision of the Manual consisted of A. J. Duncan, Chairman R. A. Freund, F. E. Grubbs, and D. C. McCune.

In the 22 years between the appearance of *ASTM STP 15D* and *Manual on Presentation of Data and Control Chart Analysis*, 6th Edition, there were two reprintings without significant changes. In that period, a number of misprints and minor inconsistencies were found in *ASTM STP 15D*. Among these were a few erroneous calculated values of control chart factors appearing in tables of **PART 3**. While all of these errors were small, the mere fact that they existed suggested a need to recalculate all tabled control chart factors. This task was carried out by A. T. A. Holden, a student at the Center for Quality and Applied Statistics at the Rochester Institute of Technology, under the general guidance of Professor E. G. Schilling of Committee E11. The tabled values of control chart factors have been corrected where found in error. In addition, some ambiguities and inconsistencies between the text and the examples on attribute control charts have received attention.

A few changes were made to bring the Manual into better agreement with contemporary statistical notation and usage. The symbol μ (Greek “mu”) has replaced \bar{X} (and \bar{X}') for the universe average of measurements (and of sample averages of those measurements). At the same time, the symbol σ has replaced σ' as the universe value of standard deviation. This entailed replacing σ by $s_{(\text{rms})}$ to denote the sample root-mean-square deviation. Replacing the universe values p' , u' , and c' by Greek letters was thought to be worse than leaving them as they are. Section 1.33, **PART 1**, on distributional information conveyed by Chebyshev’s inequality, has been revised.

Summary of changes in definitions and notations.	
MNL 7	STP 15D
μ, σ, p', u', c'	$\bar{X}, \sigma', p', u', c'$
(= universe values)	(= universe values)
$\mu_0, \sigma_0, p_0, u_0, c_0$	$\bar{X}'_0, \sigma'_0, p'_0, u'_0, c'_0$
(= standard values)	(= standard values)

In the twelve-year period since this Manual was revised again, three developments were made that had an increasing impact on the presentation of data and control chart analysis. The first was the introduction of a variety of new tools of data analysis and presentation. The effect to date of these developments is not fully reflected in **PART 1** of this edition of the Manual, but an example of the “stem and leaf” diagram is now presented in Section 15. *Manual on Presentation of Data and Control Chart Analysis*, 6th Edition from the beginning has embraced the idea that the control chart is an all-important tool for data analysis and presentation. To integrate properly the discussion of this established tool with the newer ones presents a challenge beyond the scope of this revision.

The second development of recent years strongly affecting the presentation of data and control chart analysis is the greatly increased capacity, speed, and availability of personal computers and sophisticated hand calculators. The computer revolution has not only enhanced capabilities for data analysis and presentation but also enabled techniques of high-speed real-time data-taking, analysis, and process control, which years ago would have been unfeasible, if not unthinkable. This has made it desirable to include some discussion of practical approximations for control chart factors for rapid, if not real-time, application. Supplement A has been considerably revised as a result. (The issue of approximations was raised by Professor A. L. Sweet of Purdue University.) The approximations presented in this Manual presume the computational ability to take squares and square roots of rational numbers without using tables. Accordingly, the Table of Squares and Square Roots that appeared as an Appendix to *ASTM STP 15D* was removed from the previous revision. Further discussion of approximations appears in Notes 8 and 9 of Supplement 3.B, **PART 3**. Some of the approximations presented in **PART 3** appear to be new and assume mathematical forms suggested in part by unpublished work of Dr. D. L. Jagerman of AT&T Bell Laboratories on the ratio of gamma functions with near arguments.

The third development has been the refinement of alternative forms of the control chart, especially the exponentially weighted moving average chart and the cumulative sum (“cusum”) chart. Unfortunately, time was lacking to include discussion of these developments in the fifth revision, although references are given. The assistance of S. J. Amster of AT&T Bell Laboratories in providing recent references to these developments is gratefully acknowledged.

Manual on Presentation of Data and Control Chart Analysis, 6th Edition by Committee E11 was initiated by M. G. Natrella with the help of comments from A. Bloomberg, J. T. Bygott, B. A. Drew, R. A. Freund, E. H. Jebe, B. H. Levine, D. C. McCune, R. C. Paule, R. F. Potthoff, E. G. Schilling, and R. R. Stone. The revision was completed by R. B. Murphy and R. R. Stone with further comments from A. J. Duncan, R. A. Freund, J. H. Hooper, E. H. Jebe, and T. D. Murphy.

Manual on Presentation of Data and Control Chart Analysis, 7th Edition has been directed at bringing the discussions around the various methods covered in **PART 1** up to date, especially in the areas of whole number frequency distributions,

empirical percentiles, and order statistics. As an example, an extension of the stem-and-leaf diagram has been added that is termed an “ordered stem-and-leaf,” which makes it easier to locate the quartiles of the distribution. These quartiles, along with the maximum and minimum values, are then used in the construction of a box plot.

In **PART 3**, additional material has been included to discuss the idea of risk, namely, the alpha (α) and beta (β) risks involved in the decision-making process based on data and tests for assessing evidence of nonrandom behavior in process control charts.

Also, use of the $s_{(rms)}$ statistic has been minimized in this revision in favor of the sample standard deviation s to reduce confusion as to their use. Furthermore, the graphics and tables throughout the text have been repositioned so that they appear more closely to their discussion in the text.

Manual on Presentation of Data and Control Chart Analysis, 7th Edition by Committee E11 was initiated and led by Dean V. Neubauer, Chairman of the E11.10 Subcommittee on Sampling and Data Analysis that oversees this document. Additional comments from Steve Luko, Charles Proctor, Paul Selden, Greg Gould, Frank Sinibaldi, Ray Mignogna, Neil Ullman, Thomas D. Murphy, and R. B. Murphy were instrumental in the vast majority of the revisions made in this sixth revision.

Manual on Presentation of Data and Control Chart Analysis, 8th Edition has some new material in **PART 1**. The discussion of the construction of a box plot has been supplemented with some definitions to improve clarity, and new sections have been added on probability plots and transformations.

For the first time, the manual has a new **PART 4**, which discusses material on measurement systems analysis, process capability, and process performance. This important section was deemed necessary because it is important that the measurement process be evaluated before any analysis of the process is begun. As Lord Kelvin once said: “When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge of it is of a meager and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced it to the stage of science.”

Manual on Presentation of Data and Control Chart Analysis, 8th Edition by Committee E11 was initiated and led by Dean V. Neubauer, Chairman of the E11.30 Subcommittee on Statistical Quality Control that oversees this document. Additional material from Steve Luko, Charles Proctor, and Bob Sichi, including reviewer comments from Thomas D. Murphy, Neil Ullman, and Frank Sinibaldi, were critical to the vast majority of the revisions made in this seventh revision. Thanks must also be given to Kathy Dernoga and Monica Siperko of ASTM International Publications Department for their efforts in the publication of this edition.

Foreword

This ASTM Manual on Presentation of Data and Control Chart Analysis is the eighth edition of the ASTM Manual on Presentation of Data first published in 1933. This revision was prepared by the ASTM E11.30 Subcommittee on Statistical Quality Control, which serves the ASTM Committee E11 on Quality and Statistics.

Contents

Preface	ix
PART 1: Presentation of Data	1
Summary	1
Recommendations for Presentation of Data	1
Glossary of Symbols Used in PART 1	1
Introduction	2
1.1. Purpose	2
1.2. Type of Data Considered	2
1.3. Homogeneous Data	2
1.4. Typical Examples of Physical Data	4
Ungrouped Whole Number Distribution	4
1.5. Ungrouped Distribution	4
1.6. Empirical Percentiles and Order Statistics	6
Grouped Frequency Distributions	7
1.7. Introduction	7
1.8. Definitions	7
1.9. Choice of Bin Boundaries	7
1.10. Number of Bins	7
1.11. Rules for Constructing Bins	7
1.12. Tabular Presentation	10
1.13. Graphical Presentation	10
1.14. Cumulative Frequency Distribution	10
1.15. "Stem and Leaf" Diagram	12
1.16. "Ordered Stem and Leaf" Diagram and Box Plot	12
Functions of a Frequency Distribution	13
1.17. Introduction	13
1.18. Relative Frequency	14
1.19. Average (Arithmetic Mean)	14
1.20. Other Measures of Central Tendency	14
1.21. Standard Deviation	14
1.22. Other Measures of Dispersion	14
1.23. Skewness— g_1	15
1.23a. Kurtosis— g_2	15
1.24. Computational Tutorial	15
Amount of Information Contained in p , \bar{X} , s , g_1 , and g_2	15
1.25. Summarizing the Information	15
1.26. Several Values of Relative Frequency, p	16
1.27. Single Percentile of Relative Frequency, Q_p	16
1.28. Average \bar{X} Only	16
1.29. Average \bar{X} and Standard Deviation s	17
1.30. Average \bar{X} Standard Deviation s , Skewness g_1 , and Kurtosis g_2	18
1.31. Use of Coefficient of Variation Instead of the Standard Deviation	20

1.32. General Comment on Observed Frequency Distributions of a Series of ASTM Observations	20
1.33. Summary—Amount of Information Contained in Simple Functions of the Data.	21
The Probability Plot	21
1.34. Introduction	21
1.35. Normal Distribution Case	21
1.36. Weibull Distribution Case.	23
Transformations	24
1.37. Introduction	24
1.38. Power (Variance-Stabilizing) Transformations	24
1.39. Box-Cox Transformations	24
1.40. Some Comments about the Use of Transformations	25
Essential Information	25
1.41. Introduction	25
1.42. What Functions of the Data Contain the Essential Information	25
1.43. Presenting \bar{X} Only Versus Presenting \bar{X} and s	25
1.44. Observed Relationships	26
1.45. Summary: Essential Information.	27
Presentation of Relevant Information.	27
1.46. Introduction	27
1.47. Relevant Information	27
1.48. Evidence of Control	27
Recommendations	28
1.49. Recommendations for Presentation of Data.	28
References	28
PART 2: Presenting Plus or Minus Limits of Uncertainty of an Observed Average	29
Glossary of Symbols Used in PART 2	29
2.1. Purpose	29
2.2. The Problem	29
2.3. Theoretical Background	29
2.4. Computation of Limits	30
2.5. Experimental Illustration	30
2.6. Presentation of Data	31
2.7. One-Sided Limits	32
2.8. General Comments on the Use of Confidence Limits	32
2.9. Number of Places to Be Retained in Computation and Presentation	33
Supplements	34
2.A Presenting Plus or Minus Limits of Uncertainty for σ — <i>Normal Distribution</i>	34
2.B Presenting Plus or Minus Limits of Uncertainty for p'	36
References	37
PART 3: Control Chart Method of Analysis and Presentation of Data.	38
Glossary of Terms and Symbols Used in PART 3.	38
General Principles.	39
3.1. Purpose	39
3.2. Terminology and Technical Background.	40

3.3. Two Uses	41
3.4. Breaking Up Data into Rational Subgroups	41
3.5. General Technique in Using Control Chart Method	41
3.6. Control Limits and Criteria of Control	41
Control—No Standard Given	43
3.7. Introduction	43
3.8. Control Charts for Averages \bar{X} , and for Standard Deviations, s —Large Samples	43
3.9. Control Charts for Averages \bar{X} , and for Standard Deviations, s —Small Samples	44
3.10. Control Charts for Averages \bar{X} , and for Ranges, R —Small Samples	44
3.11. Summary, Control Charts for \bar{X} , s , and R —No Standard Given	46
3.12. Control Charts for Attributes Data	46
3.13. Control Chart for Fraction Nonconforming, p	46
3.14. Control Chart for Numbers of Nonconforming Units, np	47
3.15. Control Chart for Nonconformities per Unit, u	47
3.16. Control Chart for Number of Nonconformities, c	48
3.17. Summary, Control Charts for p , np , u , and c —No Standard Given	49
Control with respect to a Given Standard	49
3.18. Introduction	49
3.19. Control Charts for Averages \bar{X} , and for Standard Deviation, s	50
3.20. Control Chart for Ranges R	50
3.21. Summary, Control Charts for \bar{X} , s , and R —Standard Given	50
3.22. Control Charts for Attributes Data	50
3.23. Control Chart for Fraction Nonconforming, p	50
3.24. Control Chart for Number of Nonconforming Units, np	52
3.25. Control Chart for Nonconformities per Unit, u	52
3.26. Control Chart for Number of Nonconformities, c	52
3.27. Summary, Control Charts for p , np , u , and c —Standard Given	53
Control Charts for Individuals	53
3.28. Introduction	53
3.29. Control Chart for Individuals, \bar{X} —Using Rational Subgroups	53
3.30. Control Chart for Individuals, \bar{X} —Using Moving Ranges	54
Examples	54
3.31. Illustrative Examples—Control, No Standard Given	54
Example 1: Control Charts for \bar{X} and s , Large Samples of Equal Size (Section 3.8A)	54
Example 2: Control Charts for \bar{X} and s , Large Samples of Unequal Size (Section 3.8B)	55
Example 3: Control Charts for \bar{X} and s , Small Samples of Equal Size (Section 3.9A)	55
Example 4: Control Charts for \bar{X} and s , Small Samples of Unequal Size (Section 3.9B)	56
Example 5: Control Charts for \bar{X} and R , Small Samples of Equal Size (Section 3.10A)	58
Example 6: Control Charts for \bar{X} and R , Small Samples of Unequal Size (Section 3.10B)	58
Example 7: Control Charts for p , Samples of Equal Size (Section 3.13A) and np , Samples of Equal Size (Section 3.14)	59
Example 8: Control Chart for p , Samples of Unequal Size (Section 3.13B)	60
Example 9: Control Charts for u , Samples of Equal Size (Section 3.15A) and c , Samples of Equal Size (Section 3.16A)	61
Example 10: Control Chart for u , Samples of Unequal Size (Section 3.15B)	62
Example 11: Control Charts for c , Samples of Equal Size (Section 3.16A)	63

3.32. Illustrative Examples—Control with Respect to a Given Standard	64
Example 12: Control Charts for \bar{X} and s , Large Samples of Equal Size (Section 3.19)	64
Example 13: Control Charts for \bar{X} and s , Large Samples of Unequal Size (Section 3.19)	65
Example 14: Control Chart for \bar{X} and s , Small Samples of Equal Size (Section 3.19)	65
Example 15: Control Chart for \bar{X} and s , Small Samples of Unequal Size (Section 3.19)	66
Example 16: Control Charts for \bar{X} and R , Small Samples of Equal Size (Sections 3.19 and 3.20)	67
Example 17: Control Charts for p , Samples of Equal Size (Section 3.23) and np , Samples of Equal Size (Section 3.24)	67
Example 18: Control Chart for p (Fraction Nonconforming), Samples of Unequal Size (Section 3.23e)	68
Example 19: Control Chart for p (Fraction Rejected), Total and Components, Samples of Unequal Size (Section 3.23)	68
Example 20: Control Chart for u , Samples of Unequal Size (Section 3.25)	71
Example 21: Control Charts for c , Samples of Equal Size (Section 3.26)	72
3.33. Illustrative Examples—Control Chart for Individuals	73
Example 22: Control Chart for Individuals, X —Using Rational Subgroups, Samples of Equal Size, No Standard Given—Based on \bar{X} and \bar{R} (Section 3.29)	73
Example 23: Control Chart for Individuals, X —Using Rational Subgroups, Standard Given, Based on μ_0 and σ_0 (Section 3.29)	74
Example 24: Control Charts for Individuals, X , and Moving Range, MR , of Two Observations, No Standard Given—Based on \bar{X} and \overline{MR} , the Mean Moving Range (Section 3.30A)	75
Example 25: Control Charts for Individuals, X , and Moving Range, MR , of Two Observations, Standard Given—Based on μ_0 and σ_0 (Section 3.30B)	76
Supplements	77
3.A Mathematical Relations and Tables of Factors for Computing Control Chart Lines	77
3.B Explanatory Notes	82
References	84
Selected Papers On Control Chart Techniques	84
PART 4: Measurements and Other Topics of Interest	86
Glossary of Terms and Symbols Used in PART 4	86
The Measurement System	87
4.1. Introduction	87
4.2. Basic Properties of a Measurement Process	87
4.3. Simple Repeatability Model	89
4.4. Simple Reproducibility	90
4.5. Measurement System Bias	90
4.6. Using Measurement Error	91
4.7. Distinct Product Categories	91
PROCESS CAPABILITY AND PERFORMANCE	92
4.8. Introduction	92
4.9. Process Capability	93
4.10. Process Capability Indices Adjusted for Process Shift, C_{pk}	94
4.11. Process Performance Analysis	94
References	95
Appendix	96
PART List of Some Related Publications on Quality Control	96
Index	97

1

Presentation of Data

PART 1 IS CONCERNED SOLELY WITH PRESENTING information about a given sample of data. It contains no discussion of inferences that might be made about the population from which the sample came.

SUMMARY

Bearing in mind that no rules can be laid down to which no exceptions can be found, the ASTM E11 committee believes that if the recommendations presented are followed, the presentations will contain the essential information for a majority of the uses made of ASTM data.

RECOMMENDATIONS FOR PRESENTATION OF DATA

Given a sample of n observations of a single variable obtained under the same essential conditions:

1. Present as a minimum, the average, the standard deviation, and the number of observations. *Always* state the number of observations.
2. Also, present the values of the maximum and minimum observations. Any collection of observations may contain mistakes. If errors occur in the collection of the data, then correct the data values, but do *not* discard or change any other observations.
3. The average and standard deviation are sufficient to describe the data, particularly so when they follow a normal distribution. To see how the data may depart from a normal distribution, prepare the grouped frequency distribution and its histogram. Also, calculate skewness, g_1 , and kurtosis, g_2 .
4. If the data seem not to be normally distributed, then one should consider presenting the median and percentiles (discussed in Section 1.6), or consider a transformation to make the distribution more normally distributed. The advice of a statistician should be sought to help determine which, if any, transformation is appropriate to suit the user's needs.
5. Present as much evidence as possible that the data were obtained under controlled conditions.
6. Present relevant information on precisely (a) the field of application within which the measurements are believed valid and (b) the conditions under which they were made.

Note

The sample proportion p is an example of a sample average in which each observation is either a 1, the occurrence of a given type, or a 0, the nonoccurrence of the same type. The sample average is then exactly the ratio, p , of the total number of occurrences to the total number possible in the sample, n .

Glossary of Symbols Used in PART 1

f	<i>Observed frequency</i> (number of observations) in a single bin of a frequency distribution
g_1	<i>Sample coefficient of skewness</i> , a measure of skewness, or lopsidedness of a distribution
g_2	<i>Sample coefficient of kurtosis</i>
n	Number of observed values (observations)
p	<i>Sample relative frequency</i> or <i>proportion</i> , the ratio of the number of occurrences of a given type to the total possible number of occurrences, the ratio of the number of observations in any stated interval to the total number of observations; <i>sample fraction nonconforming</i> for measured values the ratio of the number of observations lying outside specified limits (or beyond a specified limit) to the total number of observations
R	<i>Sample range</i> , the difference between the largest observed value and the smallest observed value
s	<i>Sample standard deviation</i>
s^2	<i>Sample variance</i>
cV	<i>Sample coefficient of variation</i> , a measure of relative dispersion based on the standard deviation (see Section 1.31)
X	Observed values of a measurable characteristic; specific observed values are designated X_1, X_2, X_3 , etc. in order of measurement, and $X_{(1)}, X_{(2)}, X_{(3)}$, etc. in order of their size, where $X_{(1)}$ is the smallest or minimum observation and $X_{(n)}$ is the largest or maximum observation in a sample of observations; also used to designate a measurable characteristic
\bar{X}	<i>Sample average</i> or <i>sample mean</i> , the sum of the n observed values in a sample divided by n

If reference is to be made to the population from which a given sample came, the following symbols should be used.

Note

If a set of data is homogeneous in the sense of Section 1.3 of **PART 1**, it is usually safe to apply statistical theory and its concepts, like that of an *expected value*, to the data to assist in its analysis and interpretation. Only then is it meaningful to speak of a population average or other characteristic relating to a population (relative) frequency distribution function of X . This function commonly assumes the form of $f(x)$, which is the probability (relative frequency) of an observation having exactly the value X , or the form of $f(x)dx$,

γ_1	Population skewness defined as the expected value (see NOTE) of $(X - \mu)^3$ divided by σ^3 . It is spelled and pronounced "gamma one."
γ_2	Population coefficient of kurtosis defined as the amount by which the expected value (see NOTE) of $(X - \mu)^4$ divided by σ^4 exceeds or falls short of 3; it is spelled and pronounced "gamma two"
μ	Population average or universe mean defined as the expected value (see NOTE) of X ; thus $E(X) = \mu$, spelled "mu" and pronounced "mew"
p'	Population relative frequency
σ	Population standard deviation, spelled and pronounced "sigma"
σ^2	Population variance defined as the expected value (see NOTE) of the square of a deviation from the universe mean; thus $E[(X - \mu)^2] = \sigma^2$
CV	Population coefficient of variation defined as the population standard deviation divided by the population mean, also called the <i>relative standard deviation</i> , or <i>relative error</i> . (see Section 1.31)

which is the probability an observation has a value between x and $x + dx$. Mathematically the *expected value* of a function of X , say $h(X)$, is defined as the sum (for discrete data) or integral (for continuous data) of that function times the probability of X and written $E[h(X)]$. For example, if the probability of X lying between x and $x + dx$ based on continuous data is $f(x)dx$, then the expected value is

$$\int h(x)f(x)dx = E[h(x)]$$

If the probability of X lying between x and $x + dx$ based on continuous data is $f(x)dx$, then the expected value is

$$\sum h(x)f(x)dx = E[h(x)].$$

Sample statistics, like \bar{X} , s^2 , g_1 , and g_2 , also have expected values in most practical cases, but these expected values relate to the population frequency distribution of *entire samples* of n observations each, rather than of individual observations. The expected value of \bar{X} is μ , the same as that of an individual observation regardless of the population frequency distribution of X , and $E(s^2) = \sigma^2$ likewise, but $E(s)$ is less than σ in all cases and its value depends on the population distribution of X .

INTRODUCTION

1.1. PURPOSE

PART 1 of the Manual discusses the application of statistical methods to the problem of: (a) condensing the information contained in a sample of observations, and (b) presenting the essential information in a concise form more readily interpretable than the unorganized mass of original data.

Attention will be directed particularly to quantitative information on measurable characteristics of materials and manufactured products. Such characteristics will be termed *quality characteristics*.

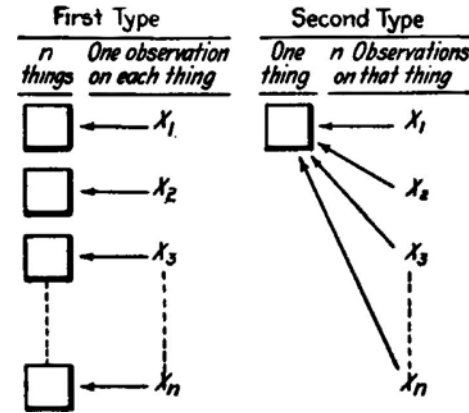


FIG. 1—Two general types of data.

1.2. TYPE OF DATA CONSIDERED

Consideration will be given to the treatment of a sample of n observations of a single variable. Figure 1 illustrates two general types: (a) the first type is a series of n observations representing single measurements of the same quality characteristic of n similar things, and (b) the second type is a series of n observations representing n measurements of the same quality characteristic of one thing.

The observations in Figure 1 are denoted as X_i , where $i = 1, 2, 3, \dots, n$. Generally, the subscript will represent the time sequence in which the observations were taken from a process or measurement. In this sense, we may consider the order of the data in Table 1 as being represented in a time-ordered manner.

Data from the first type are commonly gathered to furnish information regarding the *distribution* of the quality of the material itself, having in mind possibly some more specific purpose; such as the establishment of a quality standard or the determination of conformance with a specified quality standard, for example, 100 observations of transverse strength on 100 bricks of a given brand.

Data from the second type are commonly gathered to furnish information regarding the errors of measurement for a particular test method, for example, 50-micrometer measurements of the thickness of a test block.

Note

The quality of a material in respect to some particular characteristic, such as tensile strength, is better represented by a frequency distribution function, than by a single-valued constant.

The variability in a group of observed values of such a quality characteristic is made up of two parts: variability of the material itself, and the errors of measurement. In some practical problems, the error of measurement may be large compared with the variability of the material; in others, the converse may be true. In any case, if one is interested in discovering the objective frequency distribution of the quality of the material, consideration must be given to correcting the errors of measurement. (This is discussed in [1], pp. 379–384, in the seminal book on control chart methodology by Walter A. Shewhart.)

1.3. HOMOGENEOUS DATA

While the methods here given may be used to condense any set of observations, the results obtained by using them may be of little value from the standpoint of interpretation unless

TABLE 1—Three Groups of Original Data

(a) Transverse Strength of 270 Bricks of a Typical Brand, psi^a										
860	1,320	820	1,040	1,000	1,010	1,190	1,180	1,080	1,100	1,130
920	1,100	1,250	1,480	1,150	740	1,080	860	1,000	810	1,000
1,200	830	1,100	890	270	1,070	830	1,380	960	1,360	730
850	920	940	1,310	1,330	1,020	1,390	830	820	980	1,330
920	1,070	1,630	670	1,150	1,170	920	1,120	1,170	1,160	1,090
1,090	700	910	1,170	800	960	1,020	1,090	2,010	890	930
830	880	870	1,340	840	1,180	740	880	790	1,100	1,260
1,040	1,080	1,040	980	1,240	800	860	1,010	1,130	970	1,140
1,510	1,060	840	940	1,110	1,240	1,290	870	1,260	1,050	900
740	1,230	1,020	1,060	990	1,020	820	1,030	860	850	890
1,150	860	1,100	840	1,060	1,030	990	1,100	1,080	1,070	970
1,000	720	800	1,170	970	690	1,020	890	700	880	1,150
1,140	1,080	990	570	790	1,070	820	580	820	1,060	980
1,030	960	870	800	1,040	820	1,180	1,350	1,180	950	1,110
700	860	660	1,180	780	1,230	950	900	760	1,380	900
920	1,100	1,080	980	760	830	1,220	1,100	1,090	1,380	1,270
860	990	890	940	910	1,100	1,020	1,380	1,010	1,030	950
950	880	970	1,000	990	830	850	630	710	900	890
1,020	750	1,070	920	870	1,010	1,230	780	1,000	1,150	1,360
1,300	970	800	650	1,180	860	1,150	1,400	880	730	910
890	1,030	1,060	1,610	1,190	1,400	850	1,010	1,010	1,240	
1,080	970	960	1,180	1,050	920	1,110	780	780	1,190	
910	1,100	870	980	730	800	800	1,140	940	980	
870	970	910	830	1,030	1,050	710	890	1,010	1,120	
810	1,070	1,100	460	860	1,070	880	1,240	940	860	
(b) Weight of Coating of 100 Sheets of Galvanized Iron Sheets, oz/ft² ^b						(c) Breaking Strength of Ten Specimens of 0.104-in. Hard-Drawn Copper Wire, lb^c				
1.467	1.603	1.577	1.563	1.437		578				
1.623	1.603	1.577	1.393	1.350		572				
1.520	1.383	1.323	1.647	1.530		570				
1.767	1.730	1.620	1.620	1.383		568				
1.550	1.700	1.473	1.530	1.457		572				
1.533	1.600	1.420	1.470	1.443		570				
1.377	1.603	1.450	1.337	1.473		570				
1.373	1.477	1.337	1.580	1.433		572				
1.637	1.513	1.440	1.493	1.637		576				
1.460	1.533	1.557	1.563	1.500		584				
1.627	1.593	1.480	1.543	1.607						
1.537	1.503	1.477	1.567	1.423						

TABLE 1—Three Groups of Original Data (Continued)

(b) Weight of Coating of 100 Sheets of Galvanized Iron Sheets, oz/ft ² ^b					(c) Breaking Strength of Ten Specimens of 0.104-in. Hard-Drawn Copper Wire, lb ^c
1.533	1.600	1.550	1.670	1.573	
1.337	1.543	1.637	1.473	1.753	
1.603	1.567	1.570	1.633	1.467	
1.373	1.490	1.617	1.763	1.563	
1.457	1.550	1.477	1.573	1.503	
1.660	1.577	1.750	1.537	1.550	
1.323	1.483	1.497	1.420	1.647	
1.647	1.600	1.717	1.513	1.690	

^a Measured to the nearest 10 psi. Test method used was ASTM Method of Testing Brick and Structural Clay (C67). Data from *ASTM Manual for Interpretation of Refractory Test Data*, 1935, p. 83.

^b Measured to the nearest 0.01 oz/ft² of sheet, averaged for three spots. Test method used was ASTM Triple Spot Test of Standard Specifications for Zinc-Coated (Galvanized) Iron or Steel Sheets (A93). This has been discontinued and was replaced by ASTM Specification for General Requirements for Steel Sheet, Zinc-Coated (Galvanized) by the Hot-Dip Process (A525). Data from laboratory tests.

^c Measured to the nearest 2-lb test method used was ASTM Specification for Hard-Drawn Copper Wire (B1). Data from inspection report.

the data are good in the first place and satisfy certain requirements.

To be useful for inductive generalization, any sample of observations that is treated as a single group for presentation purposes should represent a series of measurements, all made under essentially the same test conditions, on a material or product, all of which has been produced under essentially the same conditions.

If a given sample of data consists of two or more subportions collected under different test conditions or representing material produced under different conditions, it should be considered as two or more separate subgroups of observations, each to be treated independently in the analysis. Merging of such subgroups, representing significantly different conditions, may lead to a condensed presentation that will be of little practical value. Briefly, any sample of observations to which these methods are applied should be *homogeneous*.

In the illustrative examples of **PART 1**, each sample of observations will be assumed to be homogeneous, that is, observations from a common universe of causes. The analysis and presentation by control chart methods of data obtained from several samples or capable of subdivision into subgroups on the basis of relevant engineering information is discussed in **PART 3** of this Manual. Such methods enable one to determine whether for practical purposes a given sample of observations may be considered to be homogeneous.

1.4. TYPICAL EXAMPLES OF PHYSICAL DATA

Table 1 gives three typical sets of observations, each one of these data sets represents measurements on a sample of units or specimens selected in a random manner to provide

information about the quality of a larger quantity of material—the general output of one brand of brick, a production lot of galvanized iron sheets, and a shipment of hard-drawn copper wire. Consideration will be given to ways of arranging and condensing these data into a form better adapted for practical use.

UNGROUPED WHOLE NUMBER DISTRIBUTION

1.5. UNGROUPED DISTRIBUTION

An arrangement of the observed values in ascending order of magnitude will be referred to in the Manual as the *ungrouped frequency distribution* of the data, to distinguish it from the grouped frequency distribution defined in Section 1.8. A further adjustment in the scale of the ungrouped distribution produces the whole number distribution. For example, the data from Table 1(a) were multiplied by 10^{-1} , and those of Table 1(b) by 10^3 , while those of Table 1(c) were already whole numbers. If the data carry digits past the decimal point, just round until a tie (one observation equals some other) appears and then scale to whole numbers. Table 2 presents ungrouped frequency distributions for the three sets of observations given in Table 1.

Figure 2 shows graphically the ungrouped frequency distribution of Table 2(a). In the graph, there is a minor grouping in terms of the unit of measurement. For the data from Fig. 2, it is the “rounding-off” unit of 10 psi. It is rarely desirable to present data in the manner of Table 1 or Table 2. The mind cannot grasp in its entirety the meaning of so many numbers; furthermore, greater compactness is required for most of the practical uses that are made of data.



FIG. 2—Graphically, the ungrouped frequency distribution of a set of observations. Each dot represents one brick; data are from Table 2(a).

TABLE 2—Ungrouped Frequency Distributions in Tabular Form

(a) Transverse Strength, psi [Data From Table 1(a)]										
270	780	830	870	920	970	1,020	1,070	1,100	1,180	1,310
460	780	830	880	920	980	1,020	1,070	1,100	1,180	1,320
570	780	830	880	920	980	1,020	1,070	1,100	1,180	1,330
580	790	840	880	920	980	1,020	1,070	1,100	1,180	1,330
630	790	840	880	920	980	1,020	1,070	1,110	1,180	1,340
650	800	840	880	930	980	1,020	1,070	1,110	1,180	1,350
660	800	850	880	940	980	1,020	1,070	1,110	1,180	1,360
670	800	850	890	940	990	1,030	1,080	1,120	1,190	1,360
690	800	850	890	940	990	1,030	1,080	1,120	1,190	1,380
700	800	850	890	940	990	1,030	1,080	1,130	1,190	1,380
700	800	860	890	940	990	1,030	1,080	1,130	1,200	1,380
700	800	860	890	950	990	1,030	1,080	1,140	1,220	1,380
710	810	860	890	950	1,000	1,030	1,080	1,140	1,230	1,390
710	810	860	890	950	1,000	1,040	1,080	1,140	1,230	1,400
720	820	860	890	950	1,000	1,040	1,090	1,150	1,230	1,400
730	820	860	900	960	1,000	1,040	1,090	1,150	1,240	1,480
730	820	860	900	960	1,000	1,040	1,090	1,150	1,240	1,510
730	820	860	900	960	1,000	1,050	1,090	1,150	1,240	1,610
740	820	860	900	960	1,010	1,050	1,100	1,150	1,240	1,630
740	820	860	910	970	1,010	1,050	1,100	1,150	1,250	2,010
740	820	870	910	970	1,010	1,060	1,100	1,160	1,260	
750	830	870	910	970	1,010	1,060	1,100	1,170	1,260	
760	830	870	910	970	1,010	1,060	1,100	1,170	1,270	
760	830	870	910	970	1,010	1,060	1,100	1,170	1,290	
780	830	870	920	970	1,010	1,060	1,100	1,170	1,300	
(b) Weight of Coating, oz/ft² [Data From Table 1(b)]						(c) Breaking Strength, lb [Data From Table 1(c)]				
1.323	1.457	1.513	1.567	1.620		568				
1.323	1.457	1.513	1.567	1.623		570				
1.337	1.460	1.520	1.570	1.627		570				
1.337	1.467	1.530	1.573	1.633		570				
1.337	1.467	1.530	1.573	1.637		572				
1.350	1.470	1.533	1.577	1.637		572				
1.373	1.473	1.533	1.577	1.637		572				
1.373	1.473	1.533	1.577	1.647		576				
1.377	1.473	1.537	1.580	1.647		578				
1.383	1.477	1.537	1.593	1.647		584				
1.383	1.477	1.543	1.600	1.660						
1.393	1.477	1.543	1.600	1.670						
1.420	1.480	1.550	1.600	1.690						

TABLE 2—Ungrouped Frequency Distributions in Tabular Form (Continued)

(b) Weight of Coating, oz/ft ² [Data From Table 1(b)]					(c) Breaking Strength, lb [Data From Table 1(c)]
1.420	1.483	1.550	1.603	1.700	
1.423	1.490	1.550	1.603	1.717	
1.433	1.493	1.550	1.603	1.730	
1.437	1.497	1.557	1.603	1.750	
1.440	1.500	1.563	1.607	1.753	
1.443	1.503	1.563	1.617	1.763	
1.450	1.503	1.563	1.620	1.767	

1.6. EMPIRICAL PERCENTILES AND ORDER STATISTICS

As should be apparent, the ungrouped whole number distribution may differ from the original data by a scale factor (some power of ten), by some rounding and by having been sorted from smallest to largest. These features should make it easier to convert from an ungrouped to a grouped frequency distribution. More important, they allow calculation of the *order statistics* that will aid in finding ranges of the distribution wherein lie specified proportions of the observations. A collection of observations is often seen as only a sample from a potentially huge population of observations and one aim in studying the sample may be to say what proportions of values in the population lie in certain ranges. This is done by calculating the *percentiles* of the distribution. We will see there are a number of ways to do this, but we begin by discussing order statistics and empirical estimates of percentiles.

A glance at Table 2 gives some information not readily observed in the original data set of Table 1. The data in Table 2 are arranged in increasing order of magnitude. When we arrange any data set like this, the resulting ordered sequence of values is referred to as *order statistics*. Such ordered arrangements are often of value in the initial stages of an analysis. In this context, we use subscript notation and write $X_{(i)}$ to denote the i th order statistic. For a sample of n values the order statistics are $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \dots \leq X_{(n)}$. The index i is sometimes called the *rank* of the data point to which it is attached. For a sample size of n values, the first order statistic is the smallest or minimum value and has rank 1. We write this as $X_{(1)}$. The n th order statistic is the largest or maximum value and has rank n . We write this as $X_{(n)}$. The i th order statistic is written as $X_{(i)}$, for $1 \leq i \leq n$. For the breaking strength data in Table 2c, the order statistics are: $X_{(1)} = 568$, $X_{(2)} = 570$, ..., $X_{(10)} = 584$.

When ranking the data values, we may find some that are the same. In this situation, we say that a matched set of values constitutes a *tie*. The proper rank assigned to values that make up the tie is calculated by averaging the ranks that would have been determined by the procedure above in the case where each value was different from the others. For example, there are many ties present in Table 2. Notice that $X_{(10)} = 700$, $X_{(11)} = 700$, and $X_{(12)} = 700$. Thus, the value of 700 should carry a rank equal to $(10 + 11 + 12)/3 = 11$.

The order statistics can be used for a variety of purposes, but it is for estimating the percentiles that they are used here. A *percentile* is a value that divides a distribution

to leave a given fraction of the observations less than that value. For example, the 50th percentile, typically referred to as the *median*, is a value such that half of the observations exceed it and half are below it. The 75th percentile is a value such that 25% of the observations exceed it and 75% are below it. The 90th percentile is a value such that 10% of the observations exceed it and 90% are below it.

To aid in understanding the formulas that follow, consider finding the percentile that best corresponds to a given order statistic. Although there are several answers to this question, one of the simplest is to realize that a sample of size n will partition the distribution from which it came into $n + 1$ compartments as illustrated in the following figure.

In Fig. 3, the sample size is $n = 4$; the sample values are denoted as a , b , c , and d . The sample presumably comes from some distribution as the figure suggests. Although we do not know the exact locations that the sample values correspond to along the true distribution, we observe that the four values divide the distribution into five roughly equal compartments. Each compartment will contain some percentage of the area under the curve so that the sum of each of the percentages is 100%. Assuming that each compartment contains the same area, the probability a value will fall into any compartment is $100[1/(n + 1)]\%$.

Similarly, we can compute the percentile that each value represents by $100[i/(n + 1)]\%$, where $i = 1, 2, \dots, n$. If we ask what percentile is the first order statistic among the four values, we estimate the answer as the $100[1/(4 + 1)]\% = 20\%$,

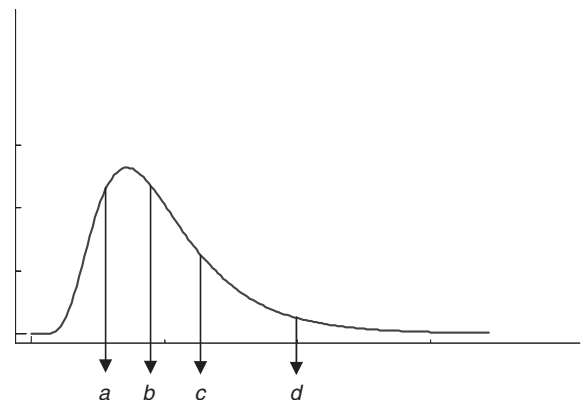


FIG. 3—Any distribution is partitioned into $n + 1$ compartments with a sample of n .

or 20th percentile. This is because, on average, each of the compartments in Figure 3 will include approximately 20% of the distribution. Since there are $n + 1 = 4 + 1 = 5$ compartments in the figure, each compartment is worth 20%. The generalization is obvious. For a sample of n values, the percentile corresponding to the i th order statistic is $100[i/(n + 1)]\%$, where $i = 1, 2, \dots, n$.

For example, if $n = 24$ and we want to know which percentiles are best represented by the 1st and 24th order statistics, we can calculate the percentile for each order statistic. For $X_{(1)}$, the percentile is $100(1)/(24 + 1) = 4$ th; and for $X_{(24)}$, the percentile is $100(24)/(24 + 1) = 96$ th. For the illustration in Figure 3, the point a corresponds to the 20th percentile, point b to the 40th percentile, point c to the 60th percentile and point d to the 80th percentile. It is not difficult to extend this application. From the figure it appears that the interval defined by $a \leq x \leq d$ should enclose, on average, 60% of the distribution of X .

We now extend these ideas to estimate the distribution percentiles. For the coating weights in Table 2(b), the sample size is $n = 100$. The estimate of the 50th percentile, or sample median, is the number lying halfway between the 50th and 51st order statistics ($X_{(50)} = 1.537$ and $X_{(51)} = 1.543$, respectively). Thus, the sample median is $(1.537 + 1.543)/2 = 1.540$. Note that the middlemost values may be the same (tie). When the sample size is an even number, the sample median will always be taken as halfway between the middle two order statistics. Thus, if the sample size is 250, the median is taken as $(X_{(125)} + X_{(126)})/2$. If the sample size is an odd number, the median is taken as the middlemost order statistic. For example, if the sample size is 13, the sample median is taken as $X_{(7)}$. Note that for an odd numbered sample size, n , the index corresponding to the median will be $i = (n + 1)/2$.

We can generalize the estimation of any percentile by using the following convention. Let p be a proportion, so that for the 50th percentile p equals 0.50, for the 25th percentile $p = 0.25$, for the 10th percentile $p = 0.10$, and so forth. To specify a percentile we need only specify p . An estimated percentile will correspond to an order statistic or weighted average of two adjacent order statistics. First, compute an approximate rank using the formula $i = (n + 1)p$. If i is an integer, then the $100p$ th percentile is estimated as $X_{(i)}$ and we are done. If i is not an integer, then drop the decimal portion and keep the integer portion of i . Let k be the retained integer portion and r be the dropped decimal portion (note: $0 < r < 1$). The estimated $100p$ th percentile is computed from the formula $X_{(k)} + r(X_{(k+1)} - X_{(k)})$.

Consider the transverse strengths with $n = 270$ and let us find the 2.5th and 97.5th percentiles. For the 2.5th percentile, $p = 0.025$. The approximate rank is computed as $i = (270 + 1) 0.025 = 6.775$. Since this is not an integer, we see that $k = 6$ and $r = 0.775$. Thus, the 2.5th percentile is estimated by $X_{(6)} + r(X_{(7)} - X_{(6)})$, which is $650 + 0.775(660 - 650) = 657.75$. For the 97.5th percentile, the approximate rank is $i = (270 + 1) 0.975 = 264.225$. Here again, i is not an integer and so we use $k = 264$ and $r = 0.225$; however, notice that both $X_{(264)}$ and $X_{(265)}$ are equal to 1,400. In this case, the value 1,400 becomes the estimate.

GROUPED FREQUENCY DISTRIBUTIONS

1.7. INTRODUCTION

Merely grouping the data values may condense the information contained in a set of observations. Such grouping involves some loss of information but is often useful in presenting engineering data. In the following sections, both tabular and graphical presentation of grouped data will be discussed.

1.8. DEFINITIONS

A grouped frequency distribution of a set of observations is an arrangement that shows the frequency of occurrence of the values of the variable in ordered classes.

The interval, along the scale of measurement, of each ordered class is termed a *bin*.

The *frequency* for any bin is the number of observations in that bin. The frequency for a bin divided by the total number of observations is the *relative frequency* for that bin.

Table 3 illustrates how the three sets of observations given in Table 1 may be organized into grouped frequency distributions. The recommended form of presenting tabular distributions is somewhat more compact, however, as shown in Table 4. Graphical presentation is used in Fig. 4 and discussed in detail in Section 1.14.

1.9. CHOICE OF BIN BOUNDARIES

It is usually advantageous to make the bin intervals equal. It is recommended that, in general, the *bin boundaries* be chosen half-way between two possible observations. By choosing bin boundaries in this way, certain difficulties of classification and computation are avoided [2, pp. 73–76]. With this choice, the bin boundary values will usually have one more significant figure (usually a 5) than the values in the original data. For example, in Table 3(a), observations were recorded to the nearest 10 psi; hence, the bin boundaries were placed at 225, 375, etc., rather than at 220, 370, etc., or 230, 380, etc. Likewise, in Table 3(b), observations were recorded to the nearest 0.01 oz/ft²; hence, bin boundaries were placed at 1.275, 1.325, etc., rather than at 1.28, 1.33, etc.

1.10. NUMBER OF BINS

The number of bins in a frequency distribution should preferably be between 13 and 20. (For a discussion of this point, see [1, p. 69] and [2, pp. 9–12].) Sturge's rule is to make the number of bins equal to $1 + 3.3\log_{10}(n)$. If the number of observations is, say, less than 250, as few as ten bins may be of use. When the number of observations is less than 25, a frequency distribution of the data is generally of little value from a presentation standpoint, as, for example, the ten observations in Table 3(c). In this case, a dot plot may be preferred over a histogram when the sample size is small, say $n < 30$. In general, the outline of a frequency distribution when presented graphically is more irregular when the number of bins is larger. This tendency is illustrated in Fig. 4.

1.11. RULES FOR CONSTRUCTING BINS

After getting the ungrouped whole number distribution, one can use a number of popular computer programs to automatically construct a histogram. For example, a spreadsheet program, such as Excel,¹ can be used by selecting the Histogram

¹ Excel is a trademark of Microsoft Corporation.

TABLE 3—Three Examples of Grouped Frequency Distribution, Showing Bin Midpoints and Bin Boundaries

	Bin Midpoint	Bin Boundaries	Observed Frequency
(a) Transverse strength, psi [data from Table 1(a)]	310	235	1
	460	385	1
	610	535	6
	760	685	45
	910	835	79
	1,060	985	79
	1,210	1,135	37
	1,350	1,285	17
	1,510	1,435	2
	1,660	1,585	2
	1,810	1,735	0
	1,960	1,885	1
		2,035	
	Total		270
(b) Weight of coating, oz/ft ² [data from Table 1(b)]	1.342	1.319.5	6
	1.387	1.364.5	6
	1.432	1.409.5	8
	1.477	1.454.5	17
	1.522	1.499.5	15
	1.567	1.544.5	17
	1.612	1.589.5	15
	1.657	1.634.5	8
	1.702	1.679.5	3
	1.747	1.724.5	5
		1.769.5	
	Total		100
(c) Breaking strength, lb [data from Table 1(c)]	567.5	565.5	1
	571.5	569.5	6
	575.5	573.5	1
	579.5	577.5	1
	583.5	581.5	1
		585.5	
	Total		10

TABLE 4—Four Methods of Presenting a Tabular Frequency Distribution [Data From Table 1(a)]			
(a) Frequency		(b) Relative Frequency (Expressed in Percentages)	
Transverse Strength, psi	Number of Bricks Having Strength within Given Limits	Transverse Strength, psi	Percentage of Bricks Having Strength within Given Limits
225 to 375	1	225 to 375	0.4
375 to 525	1	375 to 525	0.4
525 to 675	6	525 to 675	2.2
675 to 825	38	675 to 825	14.1
825 to 975	80	825 to 975	29.6
975 to 1,125	83	975 to 1,125	30.7
1,125 to 1,275	39	1,125 to 1,275	14.5
1,275 to 1,425	17	1,275 to 1,425	6.3
1,425 to 1,575	2	1,425 to 1,575	0.7
1,575 to 1,725	2	1,575 to 1,725	0.7
1,725 to 1,875	0	1,725 to 1,875	0.0
1,875 to 2,025	1	1,875 to 2,025	0.4
Total	270	Total	100.0
		Number of observations	= 270
(c) Cumulative Frequency		(d) Cumulative Relative Frequency (expressed in percentages)	
Transverse Strength, psi	Number of Bricks Having Strength Less than Given Values	Transverse Strength, psi	Percentage of Bricks Having Strength Less than Given Values
375	1	375	0.4
525	2	525	0.8
675	8	675	3.0
825	46	825	17.1
975	126	975	46.7
1,125	209	1,125	77.4
1,275	248	1,275	91.9
1,425	265	1,425	98.2
1,575	267	1,575	98.9
1,725	269	1,725	99.6
1,875	269	1,875	99.6
2,025	270	2,025	100.0
		Number of observations	= 270
Note. "Number of observations" should be recorded with tables of relative frequencies.			

item from the Analysis Toolpack menu. Alternatively, you can do it manually by applying the following rules:

- The number of bins (or "cells" or "levels") is set equal to $NL = \text{CEIL}(2.1 \ln(n))$, where n is the sample size and CEIL is an Excel spreadsheet function that extracts the largest integer part of a decimal number, e.g., 5 is CEIL(4.1)).
- Compute the bin interval as $LI = \text{CEIL}((RG + 1)/NL)$, where $RG = LW - SW$, and LW is the largest whole number and SW is the smallest among the n observations.
- Find the stretch adjustment as $SA = \text{CEIL}((NL * LI - RG)/2)$. Set the start boundary at $START = SW - SA - 0.5$ and then add LI successively NL times to get the bin

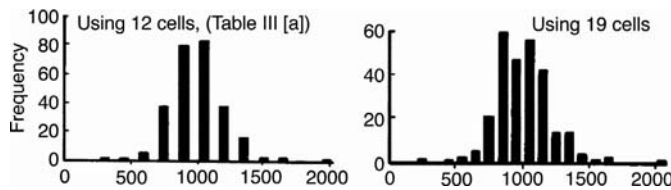


FIG. 4—Illustrations of the increased irregularity with a larger number of cells, or bins.

boundaries. Average successive pairs of boundaries to get the bin midpoints.

The data from Table 2(a) are best expressed in units of 10 psi so that, for example, 270 becomes 27. One can then verify that

- $NL = \text{CEIL}(2.1 \ln(270)) = 12$
- $RG = 201 - 27 = 174$
- $LI = \text{CEIL}(175/12) = 15$
- $SA = \text{CEIL}((180 - 174)/2) = 3$
- $\text{START} = 27 - 3 - 0.5 = 23.5$

The resulting bin boundaries with bin midpoints are shown in Table 3 for the transverse strengths.

- Having defined the bins, the last step is to count the whole numbers in each bin and thus record the grouped frequency distribution as the bin midpoints with the frequencies in each.
- The user may improve upon the rules but they will produce a useful starting point and do obey the general principles of construction of a frequency distribution.

Figure 5 illustrates a convenient method of classifying observations into bins when the number of observations is not large. For each observation, a mark is entered in the proper bin. These marks are grouped in 5's as the tallying proceeds, and the completed tabulation itself, if neatly done, provides a good picture of the frequency distribution. Notice that the bin interval has been changed from the 146 of Table 3 to a more convenient 150.

If the number of observations is, say, over 250, and accuracy is essential, the use of a computer may be preferred.

1.12. TABULAR PRESENTATION

Methods of presenting tabular frequency distributions are shown in Table 4. To make a frequency tabulation more understandable, relative frequencies may be listed as well as actual frequencies. If only relative frequencies are given, the

table cannot be regarded as complete unless the total number of observations is recorded.

Confusion often arises from failure to record bin boundaries correctly. Of the four methods, A to D, illustrated for strength measurements made to the nearest 10 lb, only methods A and B are recommended (Table 5). Method C gives no clue as to how observed values of 2,100, 2,200, etc., which fell exactly at bin boundaries were classified. If such values were consistently placed in the next higher bin, the real bin boundaries are those of method A. Method D is liable to misinterpretation since strengths were measured to the nearest 10 lb only.

1.13. GRAPHICAL PRESENTATION

Using a convenient horizontal scale for values of the variable and a vertical scale for bin frequencies, frequency distributions may be reproduced graphically in several ways as shown in Fig. 6. The *frequency bar chart* is obtained by erecting a series of bars, centered on the bin midpoints, with each bar having a height equal to the bin frequency. An alternate form of frequency bar chart may be constructed by using lines rather than bars. The distribution may also be shown by a series of points or circles representing bin frequencies plotted at bin midpoints. The *frequency polygon* is obtained by joining these points by straight lines. Each endpoint is joined to the base at the next bin midpoint to close the polygon.

Another form of graphical representation of a frequency distribution is obtained by placing along the graduated horizontal scale a series of vertical columns, each having a width equal to the bin width and a height equal to the bin frequency. Such a graph, shown at the bottom of Fig. 6, is called the *frequency histogram* of the distribution. In the histogram, if bin widths are arbitrarily given the value 1, the area enclosed by the steps represents frequency exactly, and the sides of the columns designate bin boundaries.

The same charts can be used to show relative frequencies by substituting a relative frequency scale, such as that shown in Fig. 6. It is often advantageous to show both a frequency scale and a relative frequency scale. If only a relative frequency scale is given on a chart, the number of observations should be recorded as well.

1.14. CUMULATIVE FREQUENCY DISTRIBUTION

Two methods of constructing cumulative frequency polygons are shown in Fig. 7. Points are plotted at bin boundaries.

Transverse Strength, psi.		Frequency
225 to 375		1
375 to 525		1
525 to 675		6
675 to 825		38
825 to 975		80
975 to 1,125		83
1,125 to 1,275		39
1,275 to 1,425		17
1,425 to 1,575		2
1,575 to 1,775		2
1,725 to 1,875		0
1,875 to 2,025		1
	Total	270

FIG. 5—Method of classifying observations; data from Table 1(a).

TABLE 5—Methods A through D Illustrated for Strength Measurements to the Nearest 10 lb

Recommended				Not Recommended			
Method A		Method B		Method C		Method D	
Strength, lb	Number of Observations	Strength, Lb.	Number of Observations	Strength, lb	Number of Observations	Strength, lb	Number of Observations
1,995 to 2,095	1	2,000 to 2,090	1	2,000 to 2,100	1	2,000 to 2,099	1
2,095 to 2,195	3	2,100 to 2,190	3	2,100 to 2,200	3	2,100 to 2,199	3
2,195 to 2,295	17	2,200 to 2,290	17	2,200 to 2,300	17	2,200 to 2,299	17
2,295 to 2,395	36	2,300 to 2,390	36	2,300 to 2,400	36	2,300 to 2,399	36
2,395 to 2,495	82	2,400 to 2,490	82	2,400 to 2,500	82	2,400 to 2,499	82
etc.	etc.	etc.	etc.	etc.	etc.	etc.	etc.

The upper chart gives cumulative frequency and relative cumulative frequency plotted on an arithmetic scale. This type of graph is often called an *ogive* or “s” graph. Its use is discouraged mainly because it is usually difficult to interpret the tail regions.

The lower chart shows a preferable method by plotting the relative cumulative frequencies on a normal probability scale. A normal distribution (see Fig. 14) will plot cumulatively as a straight line on this scale. Such graphs can be

drawn to show the number of observations either “less than” or “greater than” the scale values. (Graph paper with one dimension graduated in terms of the summation of normal law distribution has been described previously [4,2].) It should be noted that the cumulative percentages need to be adjusted to avoid cumulative percentages from equaling or exceeding 100%. The probability scale only reaches to 99.9% on most available probability plotting papers. Two methods that will work for estimating cumulative percentiles are [cumulative frequency/($n + 1$)] and [(cumulative frequency - 0.5)/ n].

For some purposes, the number of observations having a value “less than” or “greater than” particular scale values is

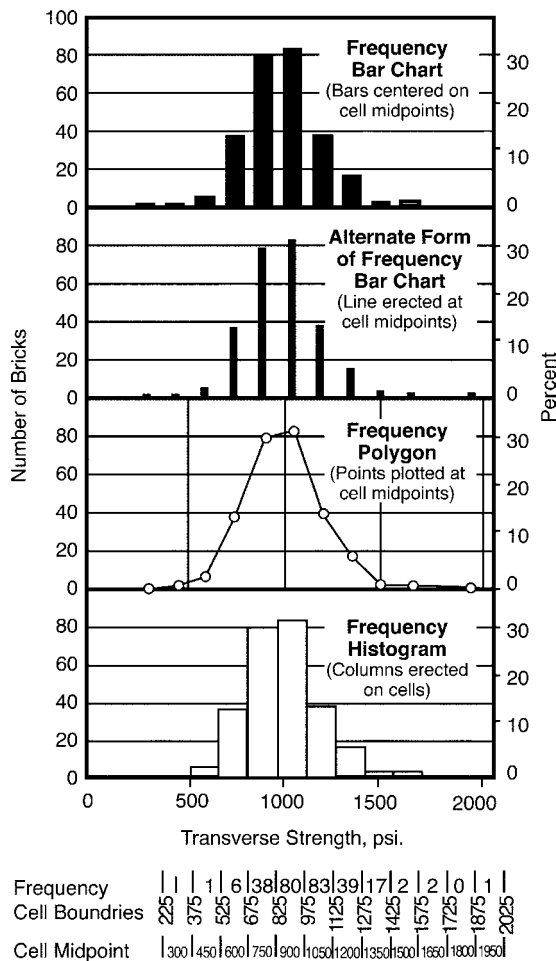
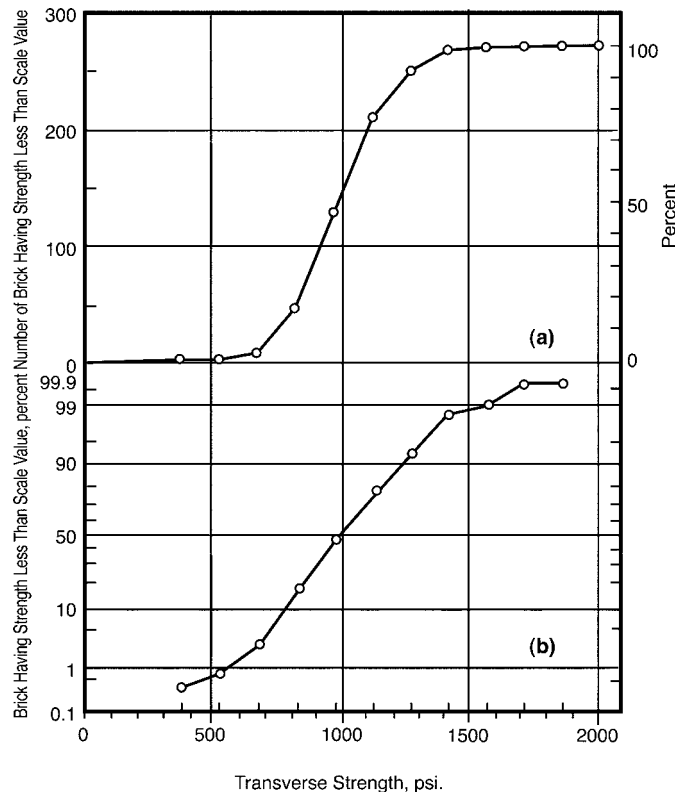


FIG. 6—Graphical presentations of a frequency distribution; data from Table 1(a) as grouped in Table 3(a).



(a) Using arithmetic scale for frequency.
(b) Using probability scale for relative frequency.

FIG. 7—Graphical presentations of a cumulative frequency distribution; data from Table 4: (a) using arithmetic scale for frequency and (b) using probability scale for relative frequency.

of more importance than the frequencies for particular bins. A table of such frequencies is termed a *cumulative frequency distribution*. The “less than” cumulative frequency distribution is formed by recording the frequency of the first bin, then the sum of the first and second bin frequencies, then the sum of the first, second, and third bin frequencies, and so on.

Because of the tendency for the grouped distribution to become irregular when the number of bins increases, it is sometimes preferable to calculate percentiles from the cumulative frequency distribution rather than from the order statistics. This is recommended as n passes the hundreds and reaches the thousands of observations. The method of calculation can easily be illustrated geometrically by using Table 4(d), Cumulative Relative Frequency, and the problem of getting the 2.5th and 97.5th percentiles.

We first define the *cumulative relative frequency function*, $F(x)$, from the bin boundaries and the cumulative relative frequencies. It is just a sequence of straight lines connecting the points $[X = 235, F(235) = 0.000]$, $[X = 385, F(385) = 0.0037]$, $[X = 535, F(535) = 0.0074]$, and so on up to $[X = 2035, F(2035) = 1.000]$. Note in Fig. 7, with an arithmetic scale for percent, that you can see the function. A horizontal line at height 0.025 will cut the curve between $X = 535$ and $X = 685$, where the curve rises from 0.0074 to 0.0296. The full vertical distance is $0.0296 - 0.0074 = 0.0222$, and the portion lacking is $0.0250 - 0.0074 = 0.0176$, so this cut will occur at $(0.0176/0.0222) 150 + 535 = 653.9$ psi. The horizontal at 97.5% cuts the curve at 1419.5 psi.

1.15. “STEM AND LEAF” DIAGRAM

It is sometimes quick and convenient to construct a “stem and leaf” diagram, which has the appearance of a histogram turned on its side. This kind of diagram does not require choosing explicit bin widths or boundaries.

The first step is to reduce the data to two or three-digit numbers by (1) dropping constant initial or final digits, like the final 0s in Table 1(a) or the initial 1s in Table 1(b); (2) removing the decimal points; and, finally, (3) rounding the results after (1) and (2), to two or three-digit numbers we can call *coded observations*. For instance, if the initial 1s and the decimal points in the data from Table 1(b) are dropped, the coded observations run from 323 to 767, spanning 445 successive integers.

If 40 successive integers per class interval are chosen for the coded observations in this example, there would be 12 intervals; if 30 successive integers, then 15 intervals; and if 20 successive integers, then 23 intervals. The choice of 12 or 23 intervals is outside of the recommended interval from 13 to 20. While either of these might nevertheless be chosen for convenience, the flexibility of the stem and leaf procedure is best shown by choosing 30 successive integers per interval, perhaps the least convenient choice of the three possibilities.

Each of the resulting 15 class intervals for the coded observations is distinguished by a first digit and a second. The third digits of the coded observations do not indicate to which intervals they belong and are therefore not needed to construct a stem and leaf diagram in this case. But the first digit may change (by 1) within a single class interval. For instance, the first class interval with coded observations beginning with 32, 33, or 34 may be identified by 3(234) and the second class interval by 3(567), but the third class interval includes coded observations with leading digits 38, 39, and 40. This interval may be identified by 3(89)4(0). The

First (and second) Digit	Second Digits Only
3(234)	3 2 2 3 3
3(567)	7 7 7 5
3(89)4(0)	8 9 8
4(123)	2 2 3 3 2
4(456)	6 6 5 5 4 5 4 6
4(789)	7 9 8 7 8 7 7 9 7 9 7 7
5(012)	2 1 0 1 0 0
5(345)	5 3 3 3 3 4 5 5 5 3 4 3 3 5
5(678)	6 7 7 7 7 6 8 6 6 7 7 6
5(9)6(01)	0 0 0 0 9 0 0 1 0
6(234)	2 3 2 4 2 3 4 2 3 3 4
6(567)	6 7
6(89)7(0)	0 9
7(123)	3 1
7(456)	6 5 6 5

FIG. 8—Stem and leaf diagram of data from Table 1(b) with groups based on triplets of first and second decimal digits.

intervals, identified in this manner, are listed in the left column of Fig. 8. Each coded observation is set down in turn to the right of its class interval identifier in the diagram using as a symbol its second digit, in the order (from left to right) in which the original observations occur in Table 1(b).

Despite the complication of changing some first digits within some class intervals, this stem and leaf diagram is quite simple to construct. In this particular case, the diagram reveals “wings” at both ends of the diagram.

As this example shows, the procedure does not require choosing a precise class interval width or boundary values. At least as important is the protection against plotting and counting errors afforded by using clear, simple numbers in the construction of the diagram—a histogram on its side. For further information on stem and leaf diagrams see [2].

1.16. “ORDERED STEM AND LEAF” DIAGRAM AND BOX PLOT

In its simplest form, a box-and-whisker plot is a method of graphically displaying the dispersion of a set of data. It is defined by the following parts:

Median divides the data set into halves; that is, 50% of the data are above the median and 50% of the data are below the median. On the plot, the median is drawn as a line cutting across the box. To determine the median, arrange the data in ascending order:

- If the number of data points is odd, the median is the middle-most point, or the $X_{((n+1)/2)}$ order statistic.
- If the number of data points is even, the average of two middle points is the median, or the average of the $X_{(n)}$ and $X_{((n+2)/2)}$ order statistics.

Lower quartile, or Q_1 , is the 25th percentile of the data. It is determined by taking the median of the lower 50% of the data.

Upper quartile, or Q_3 , is the 75th percentile of the data. It is determined by taking the median of the upper 50% of the data.

Interquartile range (IQR) is the distance between Q_3 and Q_1 . The quartiles define the box in the plot.

Whiskers are the farthest points of the data (upper and lower) not defined as outliers. Outliers are defined as any data point greater than 1.5 times the IQR away from the median. These points are typically denoted as asterisks in the plot.

First (and second) Digit	Second Digits Only
3(234)	2 2 3 3 3
3(567)	5 7 7 7
3(89)4(0)	8 8 9
4(123)	2 2 2 3 3
4(456)	4 4 5 5 5 6 6 <u>6</u>
4(789)	7 7 7 7 7 7 8 8 9 9 9
5(012)	0 0 0 1 1 2
5(345)	3 3 3 3 3 <u>3</u> 4 4 5 5 5 5
5(678)	6 6 6 6 6 7 7 7 7 7 8
5(9)6(01)	9 0 0 <u>0</u> 0 0 0 1
6(234)	2 2 2 2 3 3 3 3 4 4 4
6(567)	6 7
6(89)7(0)	9 0
7(123)	1 3
7(456)	5 5 6 6

FIG. 8a—Ordered stem and leaf diagram of data from Table 1(b) with groups based on triplets of first and second decimal digits. The 25th, 50th, and 75th quartiles are shown in bold type and are underlined.

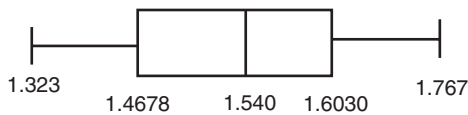


FIG. 8b—Box plot of data from Table 1(b).

The stem and leaf diagram can be extended to one that is *ordered*. The ordering pertains to the ascending sequence of values within each “leaf.” The purpose of ordering the leaves is to make the determination of the *quartiles* an easier task. The quartiles are defined above and they are found by the method discussed in Section 1.6.

In Fig. 8a, the quartiles for the data are **bold** and underlined. The quartiles are used to construct another graphic called a *box plot*.

The “box” is formed by the 25th and 75th percentiles, the center of the data is dictated by the 50th percentile (median), and “whiskers” are formed by extending a line from either side of the box to the minimum, $X_{(1)}$ point, and to the maximum, $X_{(n)}$ point. Figure 8b shows the box plot for the data from Table 1(b). For further information on box plots, see [2].

For this example, $Q_1 = 1.4678$, $Q_3 = 1.6030$, and the median = 1.540. The IQR is

$$Q_3 - Q_1 = 1.6030 - 1.4678 = 0.1352$$

which leads to a computation of the whiskers, which estimates the actual minimum and maximum values as

$$\hat{X}_{(n)} = 1.6030 + (1.5 * 0.1352) = 1.8058$$

$$\hat{X}_{(1)} = 1.4678 - (1.5 * 0.1352) = 1.2650$$

which can be compared to the actual values of 1.767 and 1.323, respectively.

The information contained in the data may also be summarized by presenting a tabular grouped frequency distribution, if the number of observations is large. A graphical presentation of a distribution makes it possible to visualize the nature and extent of the observed variation.

While some condensation is effected by presenting grouped frequency distributions, further reduction is necessary for most of the uses that are made of ASTM data. This need can be fulfilled by means of a few simple functions of the observed distribution, notably, the *average* and the *standard deviation*.

FUNCTIONS OF A FREQUENCY DISTRIBUTION

1.17. INTRODUCTION

In the problem of condensing and summarizing the information contained in the frequency distribution of a sample of observations, certain functions of the distribution are useful. For some purposes, a statement of the relative frequency within stated limits is all that is needed. For most purposes, however, two salient characteristics of the distribution that are illustrated in Fig. 9a are: (a) the position on the scale of measurement—the value about which the observations have a tendency to center, and (b) the spread or dispersion of the observations about the central value.

A third characteristic of some interest, but of less importance, is the skewness or lack of symmetry—the extent to which the observations group themselves more on one side of the central value than on the other (see Fig. 9b).

A fourth characteristic is “kurtosis,” which relates to the tendency for a distribution to have a sharp peak in the middle and excessive frequencies on the tails compared with the normal distribution or, conversely, to be relatively flat in the middle with little or no tails (see Fig. 10).

Several representative sample measures are available for describing these characteristics, but by far the most useful are the arithmetic mean \bar{X} , the standard deviation s , the skewness factor g_1 , and the kurtosis factor g_2 —all algebraic functions of the observed values. Once the numerical values of these particular measures have been determined, the original data may usually be dispensed with and two or more of these values presented instead.

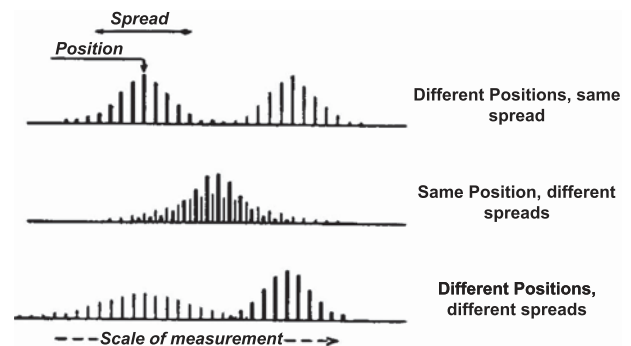


FIG. 9a—Illustration of two salient characteristics of distributions—position and spread.

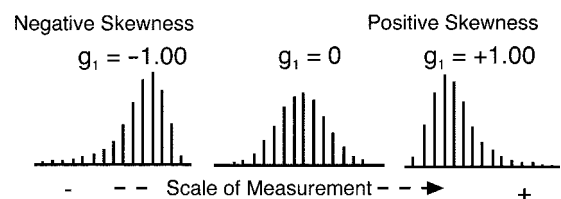


FIG. 9b—Illustration of a third characteristic of frequency distributions—skewness, and particular values of skewness, g_1 .

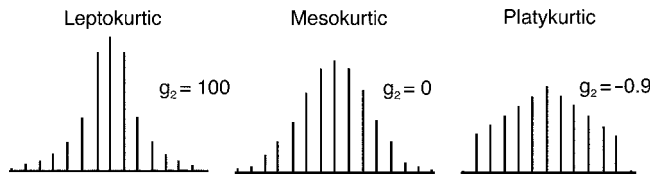


FIG. 10—Illustration of the kurtosis of a frequency distribution and particular values of g_2 .

The four characteristics of the distribution of a sample of observations just discussed are most useful when the observations form a single heap with a single peak frequency not located at either extreme of the sample values. If there is more than one peak, a tabular or graphical representation of the frequency distribution conveys information that the above four characteristics do not.

1.18. RELATIVE FREQUENCY

The relative frequency p within stated limits on the scale of measurement is the ratio of the number of observations lying within those limits to the total number of observations.

In practical work, this function has its greatest usefulness as a measure of *fraction nonconforming*, in which case it is the fraction, p , representing the ratio of the number of observations lying outside specified limits (or beyond a specified limit) to the total number of observations.

1.19. AVERAGE (ARITHMETIC MEAN)

The average (arithmetic mean) is the most widely used measure of central tendency. The term *average* and the symbol \bar{X} will be used in this Manual to represent the arithmetic mean of a sample of numbers.

The average, \bar{X} of a sample of n numbers, X_1, X_2, \dots, X_n , is the sum of the numbers divided by n , that is

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n} = \frac{\sum_{i=1}^n X_i}{n} \quad (1)$$

where the expression $\sum_{i=1}^n X_i$ means “the sum of all values of X , from X_1 to X_n , inclusive.”

Considering the n values of X as specifying the positions on a straight line of n particles of equal weight, the average corresponds to the center of gravity of the system. The average of a series of observations is expressed in the same units of measurement as the observations; that is, if the observations are in pounds, the average is in pounds.

1.20. OTHER MEASURES OF CENTRAL TENDENCY

The *geometric mean*, of a sample of n numbers, X_1, X_2, \dots, X_n , is the n th root of their product, that is

$$\text{geometric mean} = \sqrt[n]{X_1 X_2 \dots X_n} \quad (2)$$

or

log (geometric mean)

$$= \frac{\log X_1 + \log X_2 + \dots + \log X_n}{n} \quad (3)$$

Equation 1.3, obtained by taking logarithms of both sides of Eq 2, provides a convenient method for computing the geometric mean using the logarithms of the numbers.

Note

The distribution of some quality characteristics is such that a transformation, using logarithms of the observed values, gives a substantially normal distribution. When this is true, the transformation is distinctly advantageous for (in accordance with Section 1.29) much of the total information can be presented by two functions, the average, \bar{X} and the standard deviation, s , of the logarithms of the observed values. The problem of transformation is, however, a complex one that is beyond the scope of this Manual [7].

The *median* of the frequency distribution of n numbers is the middlemost value.

The *mode* of the frequency distribution of n numbers is the value that occurs most frequently. With grouped data, the mode may vary due to the choice of the interval size and the starting points of the bins.

1.21. STANDARD DEVIATION

The standard deviation is the most widely used measure of dispersion for the problems considered in **PART 1** of the Manual.

For a sample of n numbers, X_1, X_2, \dots, X_n , the sample standard deviation is commonly defined by the formula

$$s = \sqrt{\frac{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \dots + (X_n - \bar{X})^2}{n - 1}} \quad (4)$$

$$= \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n - 1}}$$

where \bar{X} is defined by Eq 1. The quantity s^2 is called the sample *variance*.

The standard deviation of any series of observations is expressed in the same units of measurement as the observations; that is, if the observations are in pounds, the standard deviation is in pounds. (Variances would be measured in pounds squared.)

A frequently more convenient formula for the computation of s is

$$s = \sqrt{\frac{\sum_{i=1}^n X_i^2 - \frac{\left(\sum_{i=1}^n X_i\right)^2}{n}}{n - 1}} \quad (5)$$

but care must be taken to avoid excessive rounding error when n is larger than s .

Note

A useful quantity related to the standard deviation is the *root-mean-square deviation*

$$s_{(rms)} = \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}} = s \sqrt{\frac{n - 1}{n}} \quad (6)$$

1.22. OTHER MEASURES OF DISPERSION

The *coefficient of variation*, cv , of a sample of n numbers, is the ratio (sometimes the coefficient is expressed as a

percentage) of their standard deviation, s , to their average \bar{X} . It is given by

$$cv = \frac{s}{\bar{X}} \quad (7)$$

The coefficient of variation is an adaptation of the standard deviation, which was developed by Prof. Karl Pearson to express the variability of a set of numbers on a relative scale rather than on an absolute scale. It is thus a dimensionless number. Sometimes it is called the *relative standard deviation*, or *relative error*.

The *average deviation* of a sample of n numbers, X_1, X_2, \dots, X_n , is the average of the absolute values of the deviations of the numbers from their average \bar{X} that is

$$\text{average deviation} = \frac{\sum_{i=1}^n |X_i - \bar{X}|}{n} \quad (8)$$

where the symbol $||$ denotes the absolute value of the quantity enclosed.

The *range* R of a sample of n numbers is the difference between the largest number and the smallest number of the sample. One computes R from the order statistics as $R = X_{(n)} - X_{(1)}$. This is the simplest measure of dispersion of a sample of observations.

1.23. SKEWNESS— g_1

A useful measure of the lopsidedness of a sample frequency distribution is the coefficient of skewness g_1 .

The coefficient of skewness g_1 , of a sample of n numbers, X_1, X_2, \dots, X_n , is defined by the expression $g_1 = k_3/s^3$. Where k_3 is the third k -statistic as defined by R. A. Fisher. The k -statistics were devised to serve as the moments of small sample data. The first moment is the mean, the second is the variance, and the third is the average of the cubed deviations and so on. Thus, $k_1 = \bar{X}$, $k_2 = s^2$,

$$k_3 = \frac{n \sum (X_i - \bar{X})^3}{(n-1)(n-2)} \quad (9)$$

Notice that when n is large

$$g_1 = \frac{\sum_{i=1}^n (X_i - \bar{X})^3}{ns^3} \quad (10)$$

This measure of skewness is a pure number and may be either positive or negative. For a symmetrical distribution, g_1 is zero. In general, for a nonsymmetrical distribution, g_1 is negative if the long tail of the distribution extends to the left, toward smaller values on the scale of measurement, and is positive if the long tail extends to the right, toward larger values on the scale of measurement. Figure 9 shows three unimodal distributions with different values of g_1 .

1.23A. KURTOSIS— g_2

The peakedness and tail excess of a sample frequency distribution are generally measured by the coefficient of kurtosis g_2 .

The coefficient of kurtosis g_2 for a sample of n numbers, X_1, X_2, \dots, X_n , is defined by the expression $g_2 = k_4/s^4$ and

$$k_4 = \frac{n(n+1) \sum (X_i - \bar{X})^4}{(n-1)(n-2)(n-3)} - \frac{3(n-1)^2 s^4}{(n-2)(n-3)} \quad (11)$$

Notice that when n is large

$$g_2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^4}{ns^4} - 3 \quad (12)$$

Again, this is a dimensionless number and may be either positive or negative. Generally, when a distribution has a sharp peak, thin shoulders, and small tails relative to the bell-shaped distribution characterized by the normal distribution, g_2 is positive. When a distribution is flat-topped with fat tails, relative to the normal distribution, g_2 is negative. Inverse relationships do not necessarily follow. We cannot definitely infer anything about the shape of a distribution from knowledge of g_2 unless we are willing to assume some theoretical curve, say a Pearson curve, as being appropriate as a graduation formula (see Fig. 14 and Section 1.30). A distribution with a positive g_2 is said to be *leptokurtic*. One with a negative g_2 is said to be *platykurtic*. A distribution with $g_2 = 0$ is said to be *mesokurtic*. Figure 10 gives three unimodal distributions with different values of g_2 .

1.24. COMPUTATIONAL TUTORIAL

The method of computation can best be illustrated with an artificial example for $n = 4$ with $X_1 = 0, X_2 = 4, X_3 = 0$, and $X_4 = 0$. First verify that $\bar{X} = 1$. The deviations from this mean are found as $-1, 3, -1$, and -1 . The sum of the squared deviations is thus 12 and $s^2 = 4$. The sum of cubed deviations is $-1 + 27 - 1 - 1 = 24$, and thus $k_3 = 16$. Now we find $g_1 = 16/8 = 2$. Verify that $g_2 = 4$. Since both g_1 and g_2 are positive, we can say that the distribution is both skewed to the right and leptokurtic relative to the normal distribution.

Of the many measures that are available for describing the salient characteristics of a sample frequency distribution, the average \bar{X} , the standard deviation s , the skewness g_1 , and the kurtosis g_2 , are particularly useful for summarizing the information contained therein. So long as one uses them only as rough indications of uncertainty we list approximate sampling standard deviations of the quantities \bar{X} , s^2 , g_1 , and g_2 , as

$$\begin{aligned} SE(\bar{X}) &= s/\sqrt{n}, \\ SE(s^2) &= s^2 \sqrt{\frac{2}{n-1}}, \\ SE(s) &= s/\sqrt{2n}, \\ SE(g_1) &= \sqrt{6/n}, \text{ and} \\ SE(g_2) &= \sqrt{24/n}, \text{ respectively} \end{aligned} \quad (13)$$

When using a computer software calculation, the ungrouped whole number distribution values will lead to less rounding off in the printed output and are simple to scale back to original units. The results for the data from Table 2 are given in Table 6.

AMOUNT OF INFORMATION CONTAINED IN p, \bar{X}, s, g_1 , AND g_2

1.25. SUMMARIZING THE INFORMATION

Given a sample of n observations, $X_1, X_2, X_3, \dots, X_n$, of some quality characteristic, how can we present concisely

TABLE 6—Summary Statistics for Three Sets of Data

Data Sets	\bar{X}	s	g_1	g_2
Transverse strength, psi	999.8	201.8	0.611	2.567
Weight of coating, oz/ft ²	1.535	0.1038	0.013	−0.291
Breaking strength, lb	573.2	4.826	1.419	1.797

information by means of which the observed distribution can be closely approximated, that is, so that the percentage of the total number, n , of observations lying within any stated interval from, say, $X = a$ to $X = b$, can be approximated?

The total information can be presented only by giving all of the observed values. It will be shown, however, that much of the total information is contained in a few simple functions—notably the average \bar{X} , the standard deviation s , the skewness g_1 , and the kurtosis g_2 .

1.26. SEVERAL VALUES OF RELATIVE FREQUENCY, p

By presenting, say, 10 to 20 values of relative frequency p , corresponding to stated bin intervals and also the number n of observations, it is possible to give practically all of the total information in the form of a tabular grouped frequency distribution. If the ungrouped distribution has any peculiarities, however, the choice of bins may have an important bearing on the amount of information lost by grouping.

1.27. SINGLE PERCENTILE OF RELATIVE FREQUENCY, Q_p

If we present but a percentile value, Q_p , of relative frequency p , such as the fraction of the total number of observed values falling outside of a specified limit and also the number n of observations, the portion of the total information presented is very small. This follows from the fact that quite dissimilar distributions may have identically the same percentile value as illustrated in Fig. 11.

Note

For the purposes of **PART 1** of this Manual, the curves of Figs. 11 and 12 may be taken to represent frequency histograms with small bin widths and based on large samples. In a frequency histogram, such as that shown at the bottom of

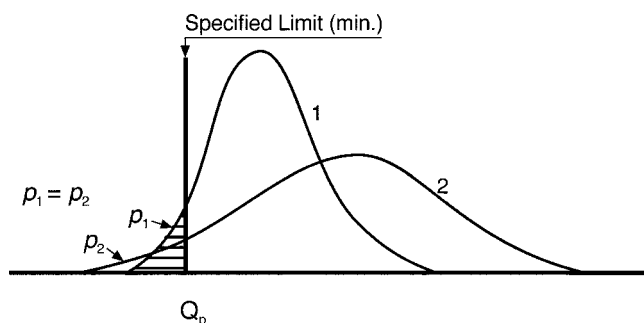


FIG. 11—Quite different distributions may have the same percentile value of p , fraction of total observations below specified limit.

Fig. 5, let the percentage relative frequency between any two bin boundaries be represented by the *area* of the histogram between those boundaries, the total area being 100%. Because the bins are of uniform width, the relative frequency in any bin is then proportional to the *height* of that bin and may be read on the vertical scale to the right.

If the sample size is increased and the bin width is reduced, a histogram in which the relative frequency is measured by area approaches as a limit the frequency distribution of the population, which in many cases can be represented by a smooth curve. The relative frequency between any two values is then represented by the *area* under the curve and between ordinates erected at those values. Because of the method of generation, the ordinate of the curve may be regarded as a curve of *relative frequency density*. This is analogous to the representation of the variation of density along a rod of uniform cross section by a smooth curve. The weight between any two points along the rod is proportional to the area under the curve between the two ordinates and we may speak of the *density* (that is, weight density) at any point but not of the *weight* at any point.

1.28. AVERAGE \bar{X} ONLY

If we present merely the average, \bar{X} and number, n , of observations, the portion of the total information presented is very small. Quite dissimilar distributions may have identically the same value of \bar{X} as illustrated in Fig. 12.

In fact, no single one of the five functions, Q_p , \bar{X} , s , g_1 , or g_2 , presented alone, is generally capable of giving much of the total information in the original distribution. Only by presenting two or three of these functions can a fairly complete description of the distribution generally be made.

An exception to the above statement occurs when theory and observation suggest that the underlying law of variation is a distribution for which the basic characteristics are all functions of the mean. For example, “life” data “under controlled conditions” sometimes follow a negative exponential distribution. For this, the cumulative relative frequency is given by the equation

$$F(X) = 1 - e^{-X/\theta} \quad 0 < X < \infty \quad (14)$$

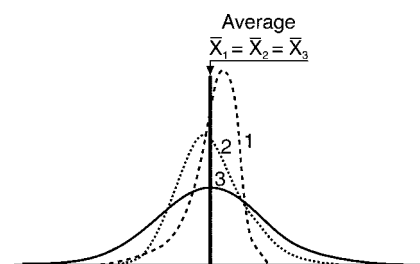


FIG. 12—Quite different distributions may have the same average.

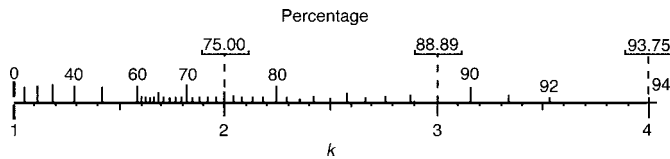


FIG. 13—Percentage of the total observations lying within the interval $\bar{x} \pm ks$ that always exceeds the percentage given on this chart.

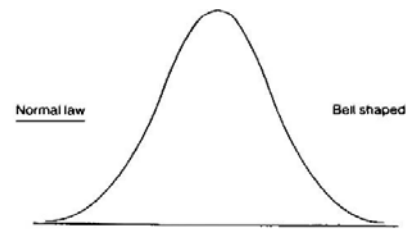
This is a single parameter distribution for which the mean and standard deviation both equal θ . That the negative exponential distribution is the underlying law of variation can be checked by noting whether values of $1 - F(X)$ for the sample data tend to plot as a straight line on ordinary semi-logarithmic paper. In such a situation, knowledge of \bar{X} will, by taking $\theta = \bar{X}$ in Eq. 14 and using tables of the exponential function, yield a fitting formula from which estimates can be made of the percentage of cases lying between any two specified values of X . Presentation of \bar{X} and n is sufficient in such cases provided they are accompanied by a statement that there are reasons to believe that X has a negative exponential distribution.

1.29. AVERAGE \bar{X} AND STANDARD DEVIATION s

These two functions contain some information even if nothing is known about the form of the observed distribution, and contain much information when certain conditions are satisfied. For example, more than $1 - 1/k^2$ of the total number n of observations lie within the closed interval $\bar{X} \pm ks$ (where k is not less than 1).

This is *Chebyshev's inequality* and is shown graphically in Fig. 13. The inequality holds true of *any* set of finite numbers regardless of how they were obtained. Thus, if \bar{X} and s are presented, we may say at once that more than 75% of the numbers lie within the interval $\bar{X} \pm 2s$; stated in another way, less than 25% of the numbers differ from \bar{X} by more than $2s$. Likewise, more than 88.9% lie within the interval $\bar{X} \pm 3s$, etc. Table 7 indicates the conformance with Chebyshev's inequality of the three sets of observations given in Table 1.

To determine approximately just what percentages of the total number of observations lie within given limits, as contrasted with minimum percentages within those limits, requires additional information of a restrictive nature. If we present \bar{X} , s , and n , and are able to add the information "data obtained under controlled conditions," then it is



Examples of two Pearson non-normal frequency curves

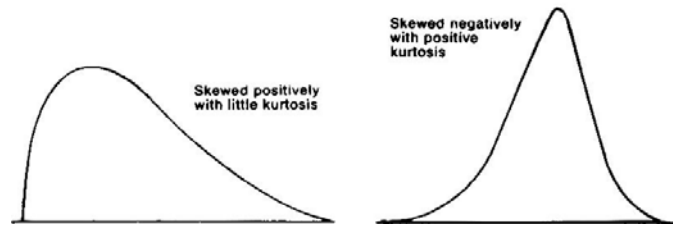


FIG. 14—A frequency distribution of observations obtained under controlled conditions will usually have an outline that conforms to the normal law or a non-normal Pearson frequency curve.

possible to make such estimates satisfactorily for limits spaced equally above and below \bar{X} .

What is meant technically by "controlled conditions" is discussed by Shewhart [1] and is beyond the scope of this Manual. Among other things, the concept of control includes the idea of homogeneous data—a set of observations resulting from measurements made under the same essential conditions and representing material produced under the same essential conditions. It is sufficient for present purposes to point out that if data are obtained under "controlled conditions," it may be assumed that the observed frequency distribution can, for most practical purposes, be graduated by some theoretical curve say, by the normal law or by one of the non-normal curves belonging to the system of frequency curves developed by Karl Pearson. (For an extended discussion of Pearson curves, see [4].) Two of these are illustrated in Fig. 14.

The applicability of the normal law rests on two converging arguments. One is mathematical and proves that the distribution of a sample mean obeys the normal law no matter what the shape of the distributions are for each of the separate observations. The other is that experience with many, many sets of data show that more of them approximate the normal law than any other distribution. In the field of statistics, this effect is known as the *central limit theorem*.

TABLE 7—Comparison of Observed Percentages and Chebyshev's Minimum Percentages of the Total Observations Lying within Given Intervals

Interval, $\bar{X} \pm ks$	Chebyshev's Minimum	Observed Percentages ^a		
	Observations Lying within the Given Interval $\bar{X} \pm ks$	Data of Table 1(a) ($n = 270$)	Data of Table 1(b) ($n = 100$)	Data of Table 1(c) ($n = 10$)
$\bar{X} \pm 2.0s$	75.0	96.7	94	90
$\bar{X} \pm 2.5s$	84.0	97.8	100	90
$\bar{X} \pm 3.0s$	88.9	98.5	100	100

^a Data from Table 1(a): $\bar{X} = 1,000$, $s = 202$; data from Table 1(b): $\bar{X} = 1.535$, $s = 0.105$; data from Table 1(c): $\bar{X} = 573.2$, $s = 4.58$.

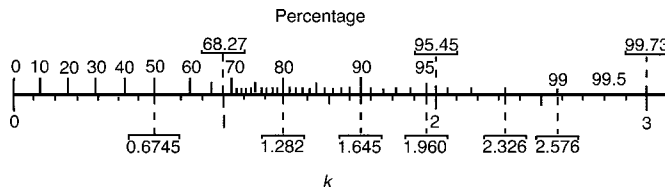


FIG. 15—Normal law integral diagram giving percentage of total area under normal law curve falling within the range $\mu \pm k\sigma$. This diagram is also useful in probability and sampling problems, expressing the upper (percentage) scale values in decimals to represent “probability.”

Supposing a smooth curve plus a gradual approach to the horizontal axis at one or both sides derived the Pearson system of curves. The normal distribution’s fit to the set of data may be checked roughly by plotting the cumulative data on normal probability paper (see Section 1.13). Sometimes if the original data do not appear to follow the normal law, some transformation of the data, such as $\log X$, will be approximately normal.

Thus, the phrase “data obtained under controlled conditions” is taken to be the equivalent of the more mathematical assertion that “the functional form of the distribution may be represented by some specific curve.” However, conformance of the shape of a frequency distribution with some curve should by no means be taken as a sufficient criterion for control.

Generally for controlled conditions, the percentage of the total observations in the original sample lying within the interval $\bar{X} \pm ks$ may be determined approximately from the chart of Fig. 15, which is based on the normal law integral. The approximation may be expected to be better the larger the number of observations. Table 8 compares the observed percentages of the total number of observations lying within several symmetrical intervals about \bar{X} with those estimated from a knowledge of \bar{X} and s , for the three sets of observations given in Table 1.

1.30. AVERAGE \bar{X} STANDARD DEVIATION s , SKEWNESS g_1 , AND KURTOSIS g_2

If the data are obtained under “controlled conditions” and if a Pearson curve is assumed appropriate as a graduation

formula, the presentation of g_1 and g_2 in addition to \bar{X} and s will contribute further information. They will give no immediate help in determining the percentage of the total observations lying within a symmetrical interval about the average \bar{X} that is, in the interval of $\bar{X} \pm ks$. What they do is to help in estimating observed percentages (in a sample already taken) in an interval whose limits are not equally spaced above and below \bar{X} .

If a Pearson curve is used as a graduation formula, some of the information given by g_1 and g_2 may be obtained from Table 9, which is taken from Table 42 of the *Biometrika Tables for Statisticians*. For $\beta_1 = g_1^2$ and $\beta_2 = g_2 + 3$, this table gives values of k_L for use in estimating the lower 2.5% of the data and values of k_U for use in estimating the upper 2.5 percentage point. More specifically, it may be estimated that 2.5% of the cases are less than $\bar{X} - k_Ls$ and 2.5% are greater than $\bar{X} + k_Us$. Put another way, it may be estimated that 95% of the cases are between $\bar{X} - k_Ls$ and $\bar{X} + k_Us$.

Table 42 of the *Biometrika Tables for Statisticians* also gives values of k_L and k_U for 0.5, 1.0, and 5.0 percentage points.

Example

For a sample of 270 observations of the transverse strength of bricks, the sample distribution is shown in Fig. 5. From the sample values of $g_1 = 0.61$ and $g_2 = 2.57$, we take $\beta_1 = g_1^2 = (0.61)^2 = 0.37$ and $\beta_2 = g_2 + 3 = 2.57 + 3 = 5.57$. Thus, from Tables 9(a) and 9(b), we may estimate that approximately 95% of the 270 cases lie between $\bar{X} - k_Ls$ and $\bar{X} + k_Us$, or between $1,000 - 1.801 (201.8) = 636.6$ and $1,000 + 2.17 (201.8) = 1,437.7$. The actual percentage of the 270 cases in this range is 96.3% [see Table 2(a)].

Notice that using just $\bar{X} \pm 1.96s$ gives the interval 604.3 to 1,395.3, which actually includes 95.9% of the cases versus a theoretical percentage of 95%. The reason we prefer the Pearson curve interval arises from knowing that the $g_1 = 0.63$ value has a standard error of 0.15 ($= \sqrt{6/270}$) and is thus about four standard errors above zero. That is, if future data come from the same conditions, it is highly probable that they will also be skewed. The 604.3 to 1,395.3 interval is symmetrical about the mean, while the 636.6 to 1,437.7 interval is offset in line with the anticipated skewness. Recall

TABLE 8—Comparison of Observed Percentages and Theoretical Estimated Percentages of the Total Observations Lying within Given Intervals

Interval, $\bar{X} \pm ks$	Theoretical Estimated Percentages ^a of Total Observations	Observed Percentages		
	Lying within the Given Interval $\bar{X} \pm ks$	Data of Table 1(a) (n = 270)	Data of Table 1(b) (n = 100)	Data of Table 1(c) (n = 10)
$\bar{X} \pm 0.6745s$	50.0	52.2	54	70
$\bar{X} \pm 1.0s$	68.3	76.3	72	80
$\bar{X} \pm 1.5s$	86.6	89.3	84	90
$\bar{X} \pm 2.0s$	95.5	96.7	94	90
$\bar{X} \pm 2.5s$	98.7	97.8	100	90
$\bar{X} \pm 3.0s$	99.7	98.5	100	100

^a Use Fig. 1.15 with \bar{X} and s as estimates of μ and σ .

TABLE 9—Lower and Upper 2.5 Percentage Points k_L and k_U of the Standardized Deviate $(X-\mu)/\sigma$, Given by Pearson Frequency Curves for Designated Values of β_1 (Estimated as Equal to g_1^2) and β_2 (Estimated as Equal to $g_2 + 3$)

	β_1/β_2	0.00	0.01	0.03	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
(a) Lower ^a k_L	1.8	1.65
	2.0	1.76	1.68	1.62	1.56
	2.2	1.83	1.76	1.71	1.66	1.57	1.49	1.41
	2.4	1.88	1.82	1.77	1.73	1.65	1.58	1.51	1.39
	2.6	1.92	1.86	1.82	1.78	1.71	1.64	1.58	1.47	1.37
	2.8	1.94	1.89	1.85	1.82	1.76	1.70	1.65	1.55	1.45	1.35
	3.0	1.96	1.91	1.87	1.84	1.79	1.74	1.69	1.60	1.52	1.42	1.33
	3.2	1.97	1.93	1.89	1.86	1.81	1.77	1.72	1.65	1.57	1.49	1.40	1.32	1.24
	3.4	1.98	1.94	1.90	1.88	1.83	1.79	1.75	1.68	1.61	1.54	1.46	1.39	1.31	1.23	...
	3.6	1.99	1.95	1.91	1.89	1.85	1.81	1.77	1.71	1.65	1.58	1.51	1.44	1.38	1.30	1.23
	3.8	1.99	1.95	1.92	1.90	1.86	1.82	1.79	1.73	1.67	1.62	1.56	1.49	1.43	1.36	1.29
	4.0	1.99	1.96	1.93	1.91	1.87	1.84	1.81	1.75	1.70	1.64	1.59	1.53	1.47	1.41	1.35
	4.2	2.00	1.96	1.93	1.91	1.88	1.84	1.82	1.76	1.72	1.67	1.62	1.56	1.51	1.45	1.40
	4.4	2.00	1.96	1.94	1.92	1.88	1.85	1.83	1.78	1.73	1.69	1.64	1.59	1.54	1.49	1.44
	4.6	2.00	1.96	1.94	1.92	1.89	1.86	1.83	1.79	1.75	1.70	1.66	1.62	1.57	1.52	1.47
	4.8	2.00	1.97	1.94	1.93	1.89	1.87	1.84	1.80	1.76	1.72	1.68	1.64	1.59	1.55	1.50
	5.0	2.00	1.97	1.94	1.93	1.90	1.87	1.85	1.81	1.77	1.73	1.69	1.65	1.61	1.57	1.53
(b) Upper k_U	1.8	1.65
	2.0	1.76	1.82	1.86	1.89
	2.2	1.83	1.89	1.93	1.96	2.00	2.04	2.06
	2.4	1.88	1.94	1.98	2.01	2.05	2.08	2.11	2.15
	2.6	1.92	1.97	2.01	2.03	2.08	2.11	2.14	2.18	2.22
	2.8	1.94	1.99	2.03	2.05	2.09	2.13	2.15	2.20	2.24	2.27
	3.0	1.96	2.01	2.04	2.06	2.10	2.13	2.16	2.21	2.25	2.28	2.32
	3.2	1.97	2.02	2.05	2.07	2.11	2.14	2.16	2.21	2.25	2.29	2.32	2.35	2.38
	3.4	1.98	2.02	2.05	2.07	2.11	2.14	2.16	2.21	2.25	2.28	2.32	2.35	2.38	2.41	...
	3.6	1.99	2.02	2.05	2.07	2.11	2.14	2.16	2.20	2.24	2.28	2.31	2.34	2.37	2.41	2.44
	3.8	1.99	2.03	2.05	2.07	2.11	2.13	2.16	2.20	2.24	2.27	2.30	2.33	2.36	2.40	2.43
	4.0	1.99	2.03	2.05	2.07	2.11	2.13	2.15	2.19	2.23	2.26	2.29	2.32	2.35	2.38	2.41
	4.2	2.00	2.03	2.05	2.07	2.10	2.13	2.15	2.19	2.22	2.25	2.28	2.31	2.34	2.37	2.40
	4.4	2.00	2.03	2.05	2.07	2.10	2.13	2.15	2.18	2.22	2.25	2.28	2.31	2.33	2.36	2.39
	4.6	2.00	2.03	2.05	2.07	2.10	2.12	2.14	2.18	2.21	2.24	2.27	2.30	2.32	2.35	2.38
	4.8	2.00	2.03	2.05	2.07	2.10	2.12	2.14	2.17	2.21	2.23	2.26	2.29	2.31	2.34	2.36
	5.0	2.00	2.03	2.05	2.07	2.09	2.12	2.14	2.17	2.20	2.23	2.25	2.28	2.30	2.33	2.35

Notes. This table was reproduced from *Biometrika Tables for Statisticians*, Vol 1, p. 207, with the kind permission of the Biometrika Trust. The Biometrika Tables also give the lower and upper 0.5, 1.0, and 5 percentage points. Use for a large sample only, say $n \geq 250$. Take $\mu = \bar{X}$ and $\sigma = s$.

^a When $g_1 > 0$, the skewness is taken to be positive, and the deviates for the lower percentage points are negative.

that the interval based on the order statistics was 657.8 to 1,400 and that from the cumulative frequency distribution was 653.9 to 1,419.5.

When computing the median, all methods will give essentially the same result but we need to choose among the methods when estimating a percentile near the extremes of the distribution.

As a first step, one should scan the data to assess its approach to the normal law. We suggest dividing g_1 and g_2 by their standard errors and, if either ratio exceeds 3, then look to see if there is an outlier. An *outlier* is an observation so small or so large that there are no other observations near it. A glance at Fig. 2 suggests the presence of outliers. This finding is reinforced by the kurtosis coefficient $g_2 = 2.567$ of Table 6 because its ratio is well above 3 at $8.6 [= 2.567/\sqrt{(24/270)}]$.

An outlier may be so extreme that persons familiar with the measurements can assert that such extreme values will not arise in the future under ordinary conditions. For example, outliers can often be traced to copying errors or reading errors or other obvious blunders. In these cases, it is good practice to discard such outliers and proceed to assess normality.

If n is very large, say $n > 10,000$, then use the percentile estimator based on the order statistics. If the ratios are both below 3, then use the normal law for smaller sample sizes. If n is between 1,000 and 10,000 but the ratios suggest skewness and/or kurtosis, then use the cumulative frequency function. For smaller sample sizes and evidence of skewness and/or kurtosis, use the Pearson system curves. Obviously, these are rough guidelines and the user must adapt them to the actual situation by trying alternative calculations and then judging the most reasonable.

Note on Tolerance Limits

In Sections 1.33 and 1.34, the percentages of X values estimated to be within a specified range pertain only to the given sample of data which is being represented succinctly by selected statistics, \bar{X} , s , etc. The Pearson curves used to derive these percentages are used simply as graduation formulas for the histogram of the sample data. The aim of Sections 1.33 and 1.34 is to indicate how much information about the sample is given by \bar{X} , s , g_1 , and g_2 . *It should be carefully noted that in an analysis of this kind the selected ranges of X and associated percentages are not to be confused with what in the statistical literature are called "tolerance limits."*

In statistical analysis, tolerance limits are values on the X scale that denote a range which may be stated to contain a specified minimum percentage of the values in the population there being attached to this statement a coefficient indicating the degree of confidence in its truth. For example, with reference to a random sample of 400 items, it may be said, with a 0.91 probability of being right, that 99% of the values in the population from which the sample came will

be in the interval $X_{(400)} - X_{(1)}$ where $X_{(400)}$ and $X_{(1)}$ are, respectively, the largest and smallest values in the sample. If the population distribution is known to be normal, it might also be said, with a 0.90 probability of being right, that 99% of the values of the population will lie in the interval $\bar{X} \pm 2.703s$. Further information on statistical tolerances of this kind is presented elsewhere [5,6,8].

1.31. USE OF COEFFICIENT OF VARIATION INSTEAD OF THE STANDARD DEVIATION

So far as quantity of information is concerned, the presentation of the sample coefficient of variation, cv , together with the average, \bar{X} , is equivalent to presenting the sample standard deviation, s , and the average, \bar{X} , because s may be computed directly from the values of $cv = s/\bar{X}$ and \bar{X} . In fact, the sample coefficient of variation (multiplied by 100) is merely the sample standard deviation, s , expressed as a percentage of the average, \bar{X} . The coefficient of variation is sometimes useful in presentations whose purpose is to compare variabilities, relative to the averages, of two or more distributions. It is also called the relative standard deviation (RSD), or relative error. The coefficient of variation should not be used over a range of values unless the standard deviation is strictly proportional to the mean within that range.

Example 1

Table 10 presents strength test results for two different materials. It can be seen that whereas the standard deviation for material B is less than the standard deviation for material A, the latter shows the greater relative variability as measured by the coefficient of variation.

The coefficient of variation is particularly applicable in reporting the results of certain measurements where the variability, σ , is known or suspected to depend on the level of the measurements. Such a situation may be encountered when it is desired to compare the variability (*a*) of physical properties of related materials usually at different levels, (*b*) of the performance of a material under two different test conditions, or (*c*) of analyses for a specific element or compound present in different concentrations.

Example 2

The performance of a material may be tested under widely different test conditions as for instance in a standard life test and in an accelerated life test. Further, the units of measurement of the accelerated life tester may be in minutes and of the standard tester, in hours. The data shown in Table 11 indicate essentially the same relative variability of performance for the two test conditions.

1.32. GENERAL COMMENT ON OBSERVED FREQUENCY DISTRIBUTIONS OF A SERIES OF ASTM OBSERVATIONS

Experience with frequency distributions for physical characteristics of materials and manufactured products prompts

TABLE 10—Strength Test Results

Material	Number of Observations, n	Average Strength, lb, \bar{X}	Standard Deviation, lb, s	Coefficient Of Variation, % cv
A	160	1,100	225	20.4
B	150	800	200	25.0

TABLE 11—Data for Two Test Conditions

Test Condition	Number of Specimens, n	Average Life, \bar{x}	Standard Deviation, s	Coefficient Of Variation, cv , %
A	50	14 h	4.2 h	30.0
B	50	80 min	23.2 min	29.0

the committee to insert a comment at this point. We have yet to find an observed frequency distribution of over 100 observations of a quality characteristic and purporting to represent essentially uniform conditions, that has less than 96% of its values within the range $\bar{X} \pm 3s$. For a normal distribution, 99.7% of the cases should theoretically lie between $\mu \pm 3\sigma$ as indicated in Fig. 15.

Taking this as a starting point and considering the fact that in ASTM work the intention is, in general, to avoid throwing together into a single series data obtained under widely different conditions—different in an important sense in respect to the characteristic under inquiry—we believe that it is possible, in general, to use the methods indicated in Sections 1.33 and 1.34 for making rough estimates of the observed percentages of a frequency distribution, at least for making estimates (per Section 1.33) for symmetrical ranges around the average, that is, $\bar{X} \pm ks$. This belief depends, to be sure, on our own experience with frequency distributions and on the observation that such distributions tend, in general, to be unimodal—to have a single peak—as in Fig. 14.

Discriminate use of these methods is, of course, presumed. The methods suggested for controlled conditions could not be expected to give satisfactory results if the parent distribution were one like that shown in Fig. 16—a bimodal distribution representing two different sets of conditions. Here, however, the methods could be applied separately to each of the two rational subgroups of data.

1.33. SUMMARY—AMOUNT OF INFORMATION CONTAINED IN SIMPLE FUNCTIONS OF THE DATA

The material given in Sections 1.24 to 1.32, inclusive, may be summarized as follows:

1. If a sample of observations of a single variable is obtained under controlled conditions, much of the total information contained therein may be made available by presenting four functions—the average \bar{X} , the standard deviation, s , the skewness, g_1 the kurtosis, g_2 and the number n , of observations. Of the four functions, \bar{X} and s contribute most; g_1 and g_2 contribute in accord with how small or how large are their standard errors, namely $\sqrt{6/n}$ and $\sqrt{24/n}$.



FIG. 16—A bimodal distribution arising from two different systems of causes.

2. The average, \bar{X} , and the standard deviation, s , give some information even for data that are not obtained under controlled conditions.
3. No single function, such as the average, of a sample of observations is capable of giving much of the total information contained therein unless the sample is from a universe that is itself characterized by a single parameter. To be confident, the population that has this characteristic will usually require much previous experience with the kind of material or phenomenon under study.

Just what functions of the data should be presented, in any instance, depends on what uses are to be made of the data. This leads to a consideration of what constitutes the “essential information.”

THE PROBABILITY PLOT

1.34. INTRODUCTION

A probability plot is a graphical device used to assess whether or not a set of data fits an assumed distribution. If a particular distribution does fit a set of data, the resulting plot may be used to estimate percentiles from the assumed distribution and even to calculate confidence bounds for those percentiles. To prepare and use a probability plot, a distribution is first assumed for the variable being studied. Important distributions that are used for this purpose include the normal, lognormal, exponential, Weibull, and extreme value distributions. In these cases, special probability “paper” is needed for each distribution. These are readily available, or their construction is available in a wide variety of software packages. The utility of a probability plot lies in the property that the sample data will generally plot as a straight line given that the assumed distribution is true. From this property, it is used as an informal and graphic hypothesis test that the sample arose from the assumed distribution. The underlying theory will be illustrated using the normal and Weibull distributions.

1.35. NORMAL DISTRIBUTION CASE

Given a sample of n observations assumed to come from a normal distribution with unknown mean and standard deviation (μ and σ), let the variable be y and the order statistics be $y_{(1)}, y_{(2)}, \dots, y_{(n)}$, see Section 1.6 for a discussion of empirical percentiles and order statistics. Associate the order statistics with certain quantiles, as described below, of the standard normal distribution. Let $\Phi(z)$ be the standard normal cumulative distribution function. Plot the order statistics, $y_{(i)}$ values, against the inverse standard normal distribution function, $z = \Phi^{-1}(p)$, evaluated at $p = i/(n + 1)$, where $i = 1, 2, 3, \dots, n$. The fraction p is referred to as the “rank at position i ” or the “plotting position at position i .” We choose this form for p because $i/(n + 1)$ is the expected fraction of a population lying below the order statistic $y_{(i)}$ in any sample of size n , from any distribution. The values for $i/(n + 1)$ are called mean ranks.

TABLE 12—List of Selected Plotting Positions

Type of Rank	Formula, p
Herd-Johnson formula (mean rank)	$i/(n + 1)$
Exact median rank	The median value of a beta distribution with parameters i and $n - i + 1$
Median rank approximation formula	$(i - 0.3)/(n + 0.4)$
Kaplan-Meier (modified)	$(i - 0.5)/n$
Modal position	$(i - 1)/(n - 1), i > 1$
Blom's approximation for a normal distribution	$(i - 0.375)/(n + 0.25)$

Several alternative rank formulas are in use. The merits of each of several commonly found rank formulas are discussed in reference [9]. In this discussion we use the mean rank $p = i/(n + 1)$ for its simplicity and ease of calculation. See the section on empirical percentiles for a graphical justification of this type of plotting position. A short table of commonly used plotting positions is shown in Table 12.

For the normal distribution, when the order statistics are plotted as described above, the resulting linear relationship is:

$$y_{(i)} = \mu + \Phi^{-1}(p)\sigma \quad (15)$$

For example, when a sample of $n = 5$ is used, the z values to use are $-0.967, -0.432, 0, 0.432$, and 0.967 . Notice that the z values will always be symmetrical because of the symmetry of the normal distribution about the mean. With the five sample values, form the ordered pairs $(y_{(i)}, z_{(i)})$ and plot these on ordinary coordinate paper. If the normal distribution assumption is true, the points will plot as an approximate straight line. The method of least squares may also be used to fit a line through the paired points [10]. When this is done, the slope of the line will approximate the standard deviation and the y intercept will approximate the mean. Such a plot is called a *normal probability plot*.

In practice, it is more common to find the y values plotted on the horizontal axis, and the cumulative probability plotted instead of the z values. With this type of plot, the vertical (probability) axis will not have a linear scale. For this practice, special normal probability paper or widely available software is in use.

Illustration 1

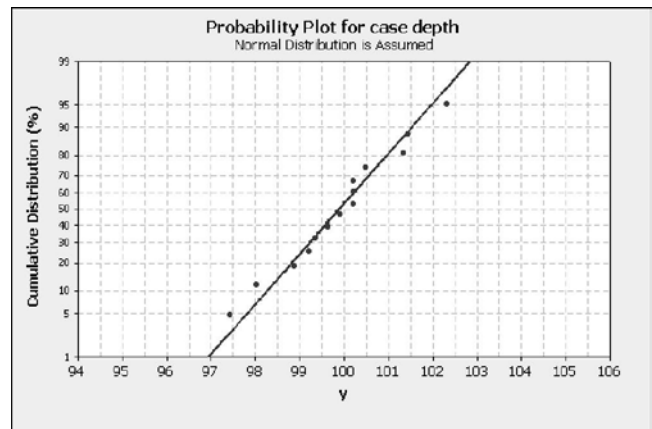
The following data are $n = 14$ case depth measurements from hardened carbide steel inserts used to secure adjoining components used in aerospace manufacture. The data are arranged with the associated steps for computing the plotting positions. Units for depth are in mills.

TABLE 13—Case Depth Data—Normal Distribution Example

y	$y_{(i)}$	i	p	$z(i)$
100.2	97.4	1	0.0667	-1.501
99.9	98.0	2	0.1333	-1.111
101.3	98.9	3	0.2000	-0.842
98.9	99.2	4	0.2667	-0.623
99.6	99.3	5	0.3333	-0.431
99.2	99.6	6	0.4000	-0.253
101.4	99.9	7	0.4667	-0.084
98.0	100.2	8	0.5333	0.084
97.4	100.2	9	0.6000	0.253
100.2	100.2	10	0.6667	0.431
102.3	100.5	11	0.7333	0.623
100.5	101.3	12	0.8000	0.842
99.3	101.4	13	0.8667	1.111
100.2	102.3	14	0.9333	1.501

In Table 13, y represents the data as obtained; $y_{(i)}$ represents the order statistics, i is the order number, $p = i/(14 + 1)$ and $z(i) = \Phi^{-1}(p)$. These data are used to create a simple type of normal probability plot. With probability paper (or using available software, such as Minitab®²), the plot generates appropriate transformations and indicates probability on the vertical axis and the variable y in the horizontal axis. Figure 17 using Minitab shows this result for the data in Table 13.

It is clear, in this case, that these data appear to follow the normal distribution. The regression of z on y would show a total sum of squares of 22.521. This is the numerator in the sample variance formula with 13 degrees of freedom. Software packages do not generally use the graphical estimate of the standard deviation for normal plots. Here we

**FIG. 17—Normal probability plot for case depth data.**

² Minitab is a registered trademark of Minitab, Inc.

use the maximum likelihood estimate of σ . In this example, this is:

$$\hat{\sigma} = \sqrt{\frac{SS_{Total}}{n}} = \sqrt{\frac{22.521}{14}} = 1.268 \quad (16)$$

1.36. WEIBULL DISTRIBUTION CASE

The probability plotting technique can be extended to several other types of distributions, most notably the Weibull distribution. In a Weibull probability plot we use the theory that the cumulative distribution function $F(x)$ is related to x through $F(x) = 1 - \exp(y/\eta)^\beta$. Here the quantities η and β are parameters of the Weibull distribution. Let $Y = \ln\{-\ln(1 - F(x))\}$. Algebraic manipulation of the equation for the Weibull distribution function $F(x)$ shows that:

$$\ln(x) = \frac{1}{\beta} Y + \ln(\eta) \quad (17)$$

For a given order statistic, $x_{(i)}$, associate an appropriate plotting position and use this in place of $F(x_{(i)})$. In practice, the approximate median rank formula $(i - 0.3)/(n + 0.4)$ is often used to estimate $F(x_{(i)})$.

Let r_i be the rank of the i th order statistic. When the distribution is Weibull, the variables $Y = \ln\{-\ln(1 - r_i)\}$ and $X = \ln(x_{(i)})$ will plot as an approximate straight line according to Eq 17. Here again, Weibull plotting paper or widely available software is required for this technique. From Eq 17, when the fitted line is obtained, the reciprocal of the slope of the line will be an estimate of the Weibull shape parameter (beta) and the scale parameter (eta) is readily estimated from the intercept term. Among “Weibull” practitioners, this technique is known as rank regression. With X and Y as defined here, it is generally agreed that the Y values have less “error” and so X on Y regression is used to obtain these estimates [10].

Illustration 2

The following data are the results of a life test of a certain type of mechanical switch. The switches were open and closed under the same conditions until failure. A sample of $n = 25$ switches were used in this test.

The data as obtained are the y values; the $y_{(i)}$ are the order statistics; i is the order number, and p is the plotting position here calculated using the approximation to the median rank, $(i - 0.3)/(n + 0.4)$. From these data X and Y coordinates, as previously defined, may be calculated. A plot of Y versus X would show a very good fit linear fit; however, we use Weibull probability paper and transform the Y coordinates to the associated probability value (plotting position). This plot is shown in Fig. 18 as generated in Minitab.

Regressing Y on X the beta parameter estimate is 6.99 and the eta parameter estimate is 20,719. These are computed using the regression results (coefficients) and the relationship to β and η in Eq 17.

The visual display of the information in a probability plot is often sufficient to judge the fit of the assumed distribution to the data. Many software packages display a “goodness of fit” statistic and associated I -value along with the plot so that the practitioner can more formally judge the fit. There are several such statistics that are used for this purpose. One of the more popular goodness of fit tests is the Anderson-Darling (AD) test. Such tests, including the AD test, are a function of the sample size and the assumed distribution. In using these tests, the hypothesis we are testing is, “The data fits the

TABLE 14—Switch Life Data—Weibull Distribution example

y	$y_{(i)}$	i	P
19,573	11,732	1	0.0275
19,008	13,897	2	0.0667
21,264	16,257	3	0.1059
17,301	16,371	4	0.1451
23,499	16,757	5	0.1843
21,103	17,301	6	0.2235
16,757	17,600	7	0.2627
20,306	17,657	8	0.3020
13,897	17,854	9	0.3412
25,341	19,008	10	0.3804
17,600	19,200	11	0.4196
22,732	19,306	12	0.4588
19,306	19,573	13	0.4980
22,776	19,940	14	0.5373
19,940	20,306	15	0.5765
22,282	20,384	16	0.6157
20,955	20,955	17	0.6549
20,384	21,103	18	0.6941
11,732	21,264	19	0.7333
17,657	22,172	20	0.7725
16,257	22,282	21	0.8118
16,371	22,732	22	0.8510
19,200	22,776	23	0.8902
17,854	23,499	24	0.9294
22,172	25,341	25	0.9686

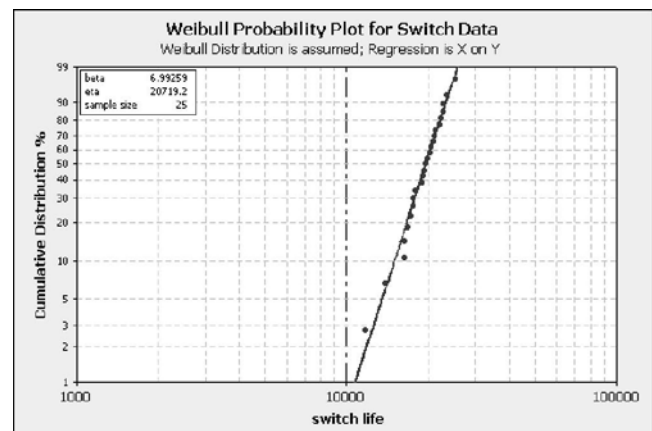


FIG. 18—Weibull probability plot of switch life data.

assumed distribution” vs. “The data do not fit.” In a hypothesis test, small P -values support our rejecting the hypothesis we are testing; therefore, in a goodness of fit test, the P -value for the test needs to be no smaller than 0.05 (or 0.10); otherwise, we have to reject the assumed distribution.

There are many reasons why a set of data will not fit a selected hypothesized distribution. The most important reason is that the data simply do not follow our assumption. In this case, we may try several different distributions. In other cases, we may have a mixture of two or more distributions; we may have outliers among our data, or we may have any number of special causes that do not allow for a good fit. In fact, the use of a probability plot often will expose such departures. In other cases, our data may fit several different distributions. In this situation, the practitioner may have to use engineering/scientific context judgment. Judgment of this type relies heavily on industry experience and perhaps some kind of expert testimony or consensus. The comparison of several P -values for a set of distributions, all of which appear to fit the data, is also a selection method in use. The distribution possessing the largest P -value is selected for use. In summary, it is typically a combination of experience, judgment and statistical methods that one uses in choosing a probability plot.

TRANSFORMATIONS

1.37. INTRODUCTION

Often, the analyst will encounter a situation where the mean of the data is correlated with its variance. The resulting distribution will typically be skewed in nature. Fortunately, if we can determine the relationship between the mean and the variance, a transformation can be selected that will result in a more symmetrical, reasonably normal, distribution for analysis.

1.38. POWER (VARIANCE-STABILIZING) TRANSFORMATIONS

An important point here is that the results of any transformation analysis pertains only to the transformed response. However, we can usually back-transform the analysis to make inferences to the original response. For example, suppose that the mean, μ , and the standard deviation, σ , are related by the following relationship:

$$\sigma \propto \mu^\alpha \quad (18)$$

The exponent of the relationship, α , can lead us to the form of the transformation needed to stabilize the variance relative to its mean. Let's say that a transformed response, Y_T , is related to its original form, Y , as

$$Y_T = Y^\lambda \quad (19)$$

The standard deviation of the transformed response will now be related to the original variable's mean, μ , by the relationship

$$\sigma_{Y_T} \propto \mu^{\lambda+\alpha-1} \quad (20)$$

In this situation, for the variance to be constant, or stabilized, the exponent must equal zero. This implies that

$$\lambda + \alpha - 1 = 0 \quad \Rightarrow \quad \lambda = 1 - \alpha \quad (21)$$

Such transformations are referred to as *power, or variance-stabilizing, transformations*. Table 15 shows some common power transformations based on α and λ .

TABLE 15—Common Power Transformations for Various Data Types

α	$\lambda = 1 - \alpha$	Transformation	Type(s) of Data
0	1	None	Normal
0.5	0.5	Square root	Poisson
1	0	Logarithm	Lognormal
1.5	−0.5	Reciprocal square root	
2	−1	Reciprocal	

Note that we could empirically determine the value for α by fitting a linear least squares line to the relationship.

$$\sigma_i \propto \mu_i^\alpha = \theta \mu_i^\alpha \quad (22)$$

which can be made linear by taking the logs of both sides of the equation, yielding

$$\log \sigma_i = \log \theta + \alpha \log \mu_i \quad (23)$$

The data take the form of the sample standard deviation, s_i , and the sample mean, \bar{X}_i , at time i . The relationship between $\log s_i$ and $\log \bar{X}_i$ can be fit with a least squares regression line. The least squares slope of the regression line is our estimate of the value of α , (see Ref. 3).

1.39. BOX-COX TRANSFORMATIONS

Another approach to determining a proper transformation is attributed to Box and Cox (see Ref. 7). Suppose that we consider our hypothetical transformation of the form in Eq 19.

$$Y_T = Y^\lambda$$

Unfortunately, this particular transformation breaks down as λ approaches 0 and Y^λ goes to 1. Transforming the data with a $\lambda = 0$ power transformation would make no sense whatsoever (all the data are equal!), so the Box-Cox procedure is discontinuous at $\lambda = 0$. The transformation takes on the following forms, depending on the value of λ :

$$Y_T = \begin{cases} (Y^\lambda - 1) / (\lambda \bar{Y}^{\lambda-1}), & \text{for } \lambda \neq 0, \\ \bar{Y} \ln Y, & \text{for } \lambda = 0 \end{cases} \quad (24)$$

where $\bar{Y} \equiv$ geometric mean of the Y_i

$$= (Y_1 Y_2 \cdots Y_n)^{1/n} \quad (25)$$

The Box-Cox procedure evaluates the change in sum of squares for error for a model with a specific value of λ . As the value of λ changes, typically between -5 and $+5$, an optimal value for the transformation occurs when the error sum of squares is minimized. This is easily seen with a plot of the SS(Error) against the value of λ .

Box-Cox plots are available in commercially available statistical programs, such as Minitab. Minitab produces a 95% (it is the default) confidence interval for lambda based on the data. Data sets will rarely produce the exact estimates of λ that are shown in Table 15. The use of a confidence interval allows the analyst to “bracket” one of the table values, so a more common transformation can be justified.

1.40. SOME COMMENTS ABOUT THE USE OF TRANSFORMATIONS

Transformations of the data to produce a more normally distributed distribution are sometimes useful, but their practical use is limited. Often the transformed data do not produce results that differ much from the analysis of the original data.

Transformations must be meaningful and, should, relate to the first principles of the problem being studied. Furthermore, according to Draper and Smith [10]:

When several sets of data arise from similar experimental situations, it may not be necessary to carry out complete analyses on all the sets to determine appropriate transformations. Quite often, the same transformation will work for all.

The fact that a general analysis exists for finding transformations does not mean that it should always be used. Often, informal plots of the data will clearly reveal the need for a transformation of an obvious kind (such as $\ln Y$ or $1/Y$). In such a case, the more formal analysis may be viewed as a useful check procedure to hold in reserve.

With respect to the use of a Box-Cox transformation, Draper and Smith offer this comment on the regression model based on a chosen λ :

The model with the “best λ ” does not guarantee a more useful model in practice. As with any regression model, it must undergo the usual checks for validity.

ESSENTIAL INFORMATION

1.41. INTRODUCTION

Presentation of data presumes some intended use either by others or by the author as supporting evidence for his or her conclusions. The objective is to present that portion of the total information given by the original data that is believed to be essential for the intended use. **Essential information** will be described as follows: “We take data to answer specific questions. We shall say that a set of statistics (functions) for a given set of data contains the **essential information** given by the data when, through the use of these statistics, we can answer the questions in such a way that further analysis of the data will not modify our answers to a practical extent” (from **PART 2** [1]).

The **Preface** to this Manual lists some of the objectives of gathering ASTM data from the type under discussion—a sample of observations of a single variable. Each such sample constitutes an observed frequency distribution, and the information contained therein should be used efficiently in answering the questions that have been raised.

1.42. WHAT FUNCTIONS OF THE DATA CONTAIN THE ESSENTIAL INFORMATION

The nature of the questions asked determine what part of the total information in the data constitutes the essential information for use in interpretation.

If we are interested in the percentages of the total number of observations that have values above (or below) several values on the scale of measurement, the essential information may be contained in a tabular grouped frequency

distribution plus a statement of the number of observations n . But even here, if n is large and if the data represent controlled conditions, the essential information may be contained in the four sample functions—the average \bar{X} , the standard deviation s , the skewness g_1 and the kurtosis g_2 , and the number of observations n . If we are interested in the average and variability of the quality of a material, or in the average quality of a material and some measure of the variability of averages for successive samples, or in a comparison of the average and variability of the quality of one material with that of other materials, or in the error of measurement of a test, or the like, then the essential information may be contained in the \bar{X} , s , and n of each sample of observations. Here, if n is small, say ten or less, much of the essential information may be contained in the \bar{X} , R (range), and n of each sample of observations. The reason for use of R when $n < 10$ is as follows:

It is important to note [11] that the expected value of the range R (largest observed value minus smallest observed value) for samples of n observations each, drawn from a normal universe having a standard deviation σ varies with sample size in the following manner.

The expected value of the range is 2.1σ for $n = 4$, 3.1σ for $n = 10$, 3.9σ for $n = 25$, and 6.1σ for $n = 500$. From this it is seen that in sampling from a normal population, the spread between the maximum and the minimum observation may be expected to be about twice as great for a sample of 25, and about three times as great for a sample of 500, as for a sample of 4. For this reason, n should *always* be given in presentations which give R . In general, it is better not to use R if n exceeds 12.

If we are also interested in the percentage of the total quantity of product that does not conform to specified limits, then part of the essential information may be contained in the observed value of fraction defective p . The conditions under which the data are obtained should always be indicated, i.e., (a) controlled, (b) uncontrolled, or (c) unknown.

If the conditions under which the data were obtained were not controlled, then the maximum and minimum observations may contain information of value.

It is to be carefully noted that if our interest goes beyond the sample data themselves to the processes that generated the samples or might generate similar samples in the future, we need to consider errors that may arise from sampling. The problems of sampling errors that arise in estimating process means, variances, and percentages are discussed in **PART 2**. For discussions of sampling errors in comparisons of means and variabilities of different samples, the reader is referred to texts on statistical theory (for example, [12]). The intention here is simply to note those statistics, those functions of the sample data, which would be useful in making such comparisons and consequently should be reported in the presentation of sample data.

1.43. PRESENTING \bar{X} ONLY VERSUS PRESENTING \bar{X} AND s

Presentation of the essential information contained in a sample of observations commonly consists in presenting \bar{X} , s , and n . Sometimes the average alone is given—no record is made of the dispersion of the observed values or of the number of observations taken. For example, Table 16 gives the observed average tensile strength for several materials under several conditions.

TABLE 16—Information of Value May Be Lost If Only the Average Is Presented

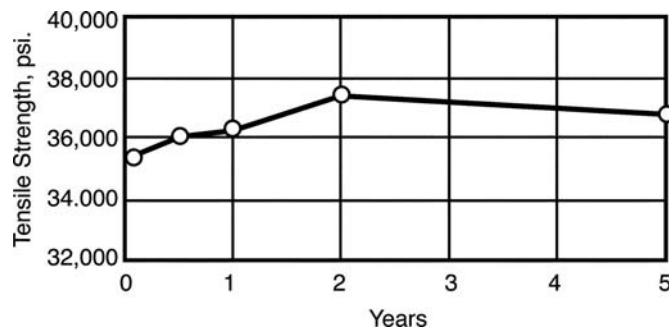
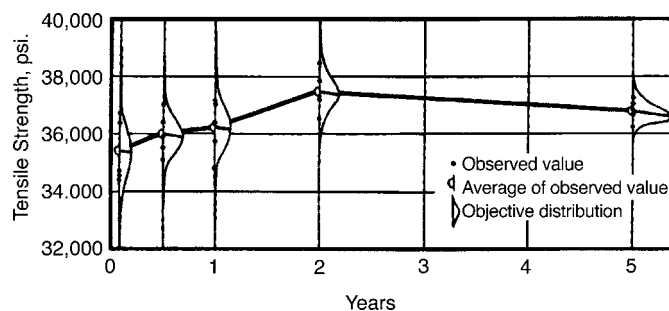
Material	Tensile Strength, psi		
	Condition a, Average, \bar{X}	Condition b, Average, \bar{X}	Condition c, Average, \bar{X}
A	51,430	47,200	49,010
B	59,060	57,380	60,700
C	57,710	74,920	80,460

The objective quality in each instance is a frequency distribution, from which the set of observed values might be considered as a sample. Presenting merely the average, and failing to present some measure of dispersion and the number of observations, generally loses much information of value. Table 17 corresponds to Table 16 and provides what will usually be considered as the essential information for several sets of observations, such as data collected in investigations conducted for the purpose of comparing the quality of different materials.

1.44. OBSERVED RELATIONSHIPS

ASTM work often requires the presentation of data showing the observed relationship between two variables. Although this subject does not fall strictly within the scope of **PART 1** of the Manual, the following material is included for general information. Attention will be given here to one type of relationship, where one of the two variables is of the nature of temperature or time—one that is controlled at will by the investigator and considered for all practical purposes as capable of “exact” measurement, free from experimental errors. (The problem of presenting information on the observed relationship between *two statistical variables*, such as hardness and tensile strength of an alloy sheet material, is more complex and will not be treated here. For further information, see [1,12,13].) Such relationships are commonly presented in the form of a chart consisting of a series of plotted points and straight lines connecting the points or a smooth curve that has been “fitted” to the points by some method or other. This section will consider merely the information associated with the plotted points, i.e., scatter diagrams.

Figure 19 gives an example of such an observed relationship. (Data are from records of shelf life tests on die-cast metals and alloys, former Subcommittee 15 of ASTM Committee B02 on Non-Ferrous Metals and Alloys.) At each

**FIG. 19—**Example of graph showing an observed relationship.**FIG. 20—**Pictorially, what lies behind the plotted points in Fig. 17. Each plotted point in Fig. 17 is the average of a sample from a universe of possible observations.

successive stage of an investigation to determine the effect of aging on several alloys, five specimens of each alloy were tested for tensile strength by each of several laboratories. The curve shows the results obtained by one laboratory for one of these alloys. Each of the plotted points is the average of five observed values of tensile strength and thus attempts to summarize an observed frequency distribution.

Figure 20 has been drawn to show pictorially what is behind the scenes. The five observations made at each stage of the life history of the alloy constitute a sample from a universe of possible values of tensile strength—an objective frequency distribution whose spread is dependent on the inherent variability of the tensile strength of the alloy and on the error of testing. The dots represent the observed values of tensile strength and the bell-shaped curves the objective distributions. In such instances, the essential information contained in the data may be made available by supplementing the graph by a tabulation of the averages, the

TABLE 17—Presentation of Essential Information (data from Table 8)

Material	Tests	Tensile Strength, psi					
		Condition a		Condition b		Condition c	
		Average, \bar{X}	Standard Deviation, s	Average, \bar{X}	Standard Deviation, s	Average, \bar{X}	Standard Deviation, s
A	20	51,430	920	47,200	830	49,010	1,070
B	18	59,060	1,320	57,380	1,360	60,700	1,480
C	27	75,710	1,840	74,920	1,650	80,460	1,910

TABLE 18—Summary of Essential Information for Fig. 20

Time of Test	Tensile Strength, psi		
	Number of Specimens	Average, \bar{X}	Standard Deviation, s
Initial	5	35,400	950
6 mo	5	35,980	668
1 yr	5	36,220	869
2 yr	5	37,460	655
5 yr	5	36,800	319

standard deviations, and the number of observations for the plotted points in the manner shown in Table 18.

1.45. SUMMARY: ESSENTIAL INFORMATION

The material given in Sections 1.41 to 1.44, inclusive, may be summarized as follows.

1. What constitutes the *essential information* in any particular instance depends on the nature of the questions to be answered, and on the nature of the hypotheses that we are willing to make based on available information. Even when measurements of a quality characteristic are made under the same essential conditions, the objective quality is a *frequency distribution* that cannot be adequately described by any single numerical value.
2. Given a series of observations of a single variable arising from the same essential conditions, it is the opinion of the committee that the average, \bar{X} , the standard deviation, s , and the number, n , of observations contain the essential information for a majority of the uses made of such data in ASTM work.

Note

If the observations are not obtained under the same essential conditions, analysis and presentation by the control chart method, in which *order* (see **PART 3** of this Manual) is taken into account by rational subgrouping of observations, commonly provide important additional information.

PRESENTATION OF RELEVANT INFORMATION

1.46. INTRODUCTION

Empirical knowledge is not contained in the observed data alone; rather it arises from interpretation—an act of thought. (For an important discussion on the significance of prior information and hypothesis in the interpretation of data, see [14]; a treatise on the philosophy of probable inference that is of basic importance in the interpretation of any and all data is presented [15].) Interpretation consists in testing hypotheses based on prior knowledge. Data constitute but a part of the information used in interpretation—the judgments that are made depend as well on pertinent collateral information, much of which may be of a qualitative rather than of a quantitative nature.

If the data are to furnish a basis for most valid prediction, they must be obtained under controlled conditions and must be free from constant errors of measurement. Mere presentation does not alter the goodness or badness of data.

However, the usefulness of good data may be enhanced by the manner in which they are presented.

1.47. RELEVANT INFORMATION

Presented data should be accompanied by any or all available relevant information, particularly information on *precisely* the field within which the measurements are supposed to hold and the condition under which they were made, and evidence that the data are good. Among the specific things that may be presented with ASTM data to assist others in interpreting them or to build up confidence in the interpretation made by an author are:

1. The kind, grade, and character of material or product tested.
2. The mode and conditions of production, if this has a bearing on the feature under inquiry.
3. The method of selecting the sample; steps taken to ensure its randomness or representativeness. (The manner in which the sample is taken has an important bearing on the interpretability of data and is discussed by Dodge [16].)
4. The specific method of test (if an ASTM or other standard test, so state, together with any modifications of procedure).
5. The specific conditions of test, particularly the regulation of factors that are known to have an influence on the feature under inquiry.
6. The precautions or steps taken to eliminate systematic or constant errors of observation.
7. The difficulties encountered and eliminated during the investigation.
8. Information regarding parallel but independent paths of approach to the end results.
9. Evidence that the data were obtained under controlled conditions; the results of statistical tests made to support belief in the constancy of conditions, in respect to the physical tests made or the material tested, or both. (Here, we mean constancy in the statistical sense, which encompasses the thought of stability of conditions from one time to another and from one place to another. This state of affairs is commonly referred to as “statistical control.” Statistical criteria have been developed by means of which we may judge when controlled conditions exist. Their character and mode of application are given in **PART 3** of this Manual; see also [17].)

Much of this information may be qualitative in character, and some may even be vague, yet without it, the interpretation of the data and the conclusions reached may be misleading or of little value to others.

1.48. EVIDENCE OF CONTROL

One of the fundamental requirements of good data is that they should be obtained under controlled conditions. The interpretation of the observed results of an investigation depends on whether there is justification for believing that the conditions were controlled.

If the data are numerous and statistical tests for control are made, evidence of control may be presented by giving the results of these tests. (For examples, see [18–21].) Such quantitative evidence greatly strengthens inductive arguments. In any case, it is important to indicate clearly just what precautions were taken to control the essential conditions. Without tangible evidence of this character, the

reader's degree of rational belief in the results presented will depend on his faith in the ability of the investigator to eliminate all causes of lack of constancy.

RECOMMENDATIONS

1.49. RECOMMENDATIONS FOR PRESENTATION OF DATA

The following recommendations for presentation of data apply for the case where one has at hand a sample of n observations of a single variable obtained under the same essential conditions.

1. Present as a minimum, the average, the standard deviation, and the number of observations. *Always* state the number of observations taken.
2. If the number of observations is moderately large ($n > 30$), present also the value of the skewness, g_1 , and the value of the kurtosis, g_2 . An additional procedure when n is large ($n > 100$) is to present a graphical representation, such as a grouped frequency distribution.
3. If the data were not obtained under controlled conditions and it is desired to give information regarding the extreme observed effects of assignable causes, present the values of the maximum and minimum observations in addition to the average, the standard deviation, and the number of observations.
4. Present as much evidence as possible that the data were obtained under controlled conditions.
5. Present relevant information on precisely (a) the field within which the measurements are believed valid and (b) the conditions under which they were made.

References

- [1] Shewhart, W.A., *Economic Control of Quality of Manufactured Product*, Van Nostrand, New York, 1931; republished by ASQC Quality Press, Milwaukee, WI, 1980.
- [2] Tukey, J.W., *Exploratory Data Analysis*, Addison-Wesley, Reading, PA, 1977, pp. 1–26.
- [3] Box, G.E.P., Hunter, W.G. and Hunter, J.S., *Statistics for Experimenters*, Wiley, New York, 1978, pp. 329–330.
- [4] Elderton, W.P. and Johnson, N.L., *Systems of Frequency Curves*, Cambridge University Press, Bentley House, London, 1969.
- [5] Duncan, A.J., *Quality Control and Industrial Statistics*, 5th ed., Chapter 6, Sections 4 and 5, Richard D. Irwin, Inc., Homewood, IL, 1986.
- [6] Bowker, A.H. and Lieberman, G.J., *Engineering Statistics*, 2nd ed., Section 8.12, Prentice-Hall, New York, 1972.
- [7] Box, G.E.P. and Cox, D.R., "An Analysis of Transformations," *J. R. Stat. Soc. B*, Vol. 26, 1964, pp. 211–243.
- [8] Ott, E.R., Schilling, E.G. and Neubauer, D.V., *Process Quality Control*, 4th ed., McGraw-Hill, New York, 2005.
- [9] Hyndman, R.J. and Fan, Y., "Sample Quantiles in Statistical Packages," *Am. Stat.*, Vol. 50, 1996, pp. 361–365.
- [10] Draper, N.R. and Smith, H., *Applied Regression Analysis*, 3rd ed., John Wiley & Sons, Inc., New York, 1998, p. 279.
- [11] Tippett, L.H.C., "On the Extreme Individuals and the Range of Samples Taken from a Normal Population," *Biometrika*, Vol. 17, Dec. 1925, pp. 364–387.
- [12] Hoel, P.G., *Introduction to Mathematical Statistics*, 5th ed., Wiley, New York, 1984.
- [13] Yule, G.U. and Kendall, M.G., *An Introduction to the Theory of Statistics*, 14th ed., Charles Griffin and Company, Ltd., London, 1950.
- [14] Lewis, C.I., *Mind and the World Order*, Scribner, New York, 1929.
- [15] Keynes, J.M., *A Treatise on Probability*, MacMillan, New York, 1921.
- [16] Dodge, H.F., "Statistical Control in Sampling Inspection," presented at a Round Table Discussion on "Acquisition of Good Data," held at the 1932 Annual Meeting of the ASTM International; published in *American Machinist*, Oct. 26 and Nov. 9, 1932.
- [17] Pearson, E.S., "A Survey of the Uses of Statistical Method in the Control and Standardization of the Quality of Manufactured Products," *J. R. Stat. Soc.*, Vol. XCVI, Part 11, 1933, pp. 21–60.
- [18] Passano, R.F., "Controlled Data from an Immersion Test," *Proceedings*, ASTM International, West Conshohocken, PA, Vol. 32, Part 2, 1932, p. 468.
- [19] Skinker, M.F., "Application of Control Analysis to the Quality of Varnished Cambric Tape," *Proceedings*, ASTM International, West Conshohocken, PA, Vol. 32, Part 3, 1932, p. 670.
- [20] Passano, R.F. and Nagley, F.R., "Consistent Data Showing the Influences of Water Velocity and Time on the Corrosion of Iron," *Proceedings*, ASTM International, West Conshohocken, PA, Vol. 33, Part 2, p. 387.
- [21] Chancellor, W.C., "Application of Statistical Methods to the Solution of Metallurgical Problems in Steel Plant," *Proceedings*, ASTM International, West Conshohocken, PA, Vol. 34, Part 2, 1934, p. 891.

2

Presenting Plus or Minus Limits of Uncertainty of an Observed Average

Glossary of Symbols Used in PART 2

μ	Population mean
a	Factor, given in Table 2 of PART 2 , for computing confidence limits for μ associated with a desired value of probability, P , and a given number of observations, n
k	Deviation of a normal variable
n	Number of observed values (observations)
p	Sample fraction nonconforming
p'	Population fraction nonconforming
σ	Population standard deviation
P	Probability; used in PART 2 to designate the probability associated with confidence limits; relative frequency with which the averages μ of sampled populations may be expected to be included within the confidence limits (for μ) computed from samples
s	Sample standard deviation
$\hat{\sigma}$	Estimate of σ based on several samples
X	Observed value of a measurable characteristic; specific observed values are designated X_1, X_2, X_3 , etc.; also used to designate a measurable characteristic
\bar{X}	<i>Sample average</i> (arithmetic mean), the sum of the n observed values in a set divided by n

2.1. PURPOSE

PART 2 of the Manual discusses the problem of presenting plus or minus limits to indicate the uncertainty of the average of a number of observations obtained under the same essential conditions and suggests a form of presentation for use in ASTM reports and publications where needed.

2.2. THE PROBLEM

An observed average, \bar{X} , is subject to the uncertainties that arise from sampling fluctuations and tends to differ from the population mean. The smaller the number of observations, n , the larger the number of fluctuations is likely to be.

With a set of n observed values of a variable X , whose average (arithmetic mean) is \bar{X} , as in Table 1, it is often desired to interpret the results in some way. One way is to construct an interval such that the mean, $\mu = 573.2 \pm 3.5$ lb, lies within limits being established from the quantitative data along with the implications that the mean μ of the population sampled is included within these limits with a specified probability. How

should such limits be computed, and what meaning may be attached to them?

Note

The mean μ is the value of \bar{X} that would be approached as a statistical limit as more and more observations were obtained under the same essential conditions and their cumulative averages were computed.

2.3. THEORETICAL BACKGROUND

Mention should be made of the practice, now mostly out of date in scientific work, of recording such limits as

$$\bar{X} \pm 0.6745 \frac{s}{\sqrt{n}}$$

where

\bar{X} = observed average,

s = observed standard deviation, and

n = number of observations,

and referring to the value $0.6745 s/\sqrt{n}$ as the “probable error” of the observed average, \bar{X} . (Here the value of 0.6745 corresponds to the normal law probability of 0.50; see Table 8 of **PART 1**.) The term “probable error” and the probability value of 0.50 properly apply to the errors of sampling when sampling from a universe whose average, μ , and whose standard deviation, σ , are *known* (these terms apply to limits $\mu \pm 0.6745 \sigma/\sqrt{n}$), but they do not apply in the inverse problem when merely sample values of \bar{X} and s are given.

Investigation of this problem [1–3] has given a more satisfactory alternative (Section 2.4), a procedure that provides limits that have a definite operational meaning.

Note

While the method of Section 2.4 represents the best that can be done at present in interpreting a sample \bar{X} and s , when no other information regarding the variability of the population is available, a much more satisfactory interpretation can be made in general if other information regarding the variability of the population is at hand, such as a series of samples from the universe or similar populations for each of which a value of s or R is computed. If s or R displays statistical control, as outlined in **PART 3** of this Manual, and a sufficient number of samples (preferably 20 or more) are available to obtain a reasonably precise estimate of σ , designated as $\hat{\sigma}$, the limits of uncertainty for a sample containing any number of observations, n , and arising from a population

TABLE 1—Breaking Strength of Ten Specimens of 0.104-in. Hard-Drawn Copper Wire

Specimen	Breaking Strength, X , lb
1	578
2	572
3	570
4	568
5	572
6	570
7	570
8	572
9	576
10	584
$n = 10$	5,732
Average, \bar{X}	573.2
Standard deviation, s	4.83

whose true standard deviation can be presumed to be equal to $\hat{\sigma}$, can be computed from the following formula:

$$\bar{X} \pm k \frac{\hat{\sigma}}{\sqrt{n}}$$

where $k = 1.645, 1.960$, and 2.576 for probabilities of $P = 0.90, 0.95$, and 0.99 , respectively.

2.4. COMPUTATION OF LIMITS

The following procedure applies to any long-run *series* of problems for *each* of which the following conditions are met:

GIVEN

A sample of n observations, $X_1, X_2, X_3, \dots, X_n$, having an average $= \bar{X}$ and a standard deviation $= s$.

CONDITIONS

(a) The population sampled is homogeneous (statistically controlled) in respect to X , the variable measured. (b) The distribution of X for the population sampled is approximately normal. (c) The sample is a random sample.¹

Procedure: Compute limits

$$\bar{X} \pm as$$

where the value of a is given in Table 2 for three values of P and for various values of n .

MEANING

If the values of a given in Table 2 for $P = 0.95$ are used in a series of such problems, then, in the long run, we may

expect 95 % of the intervals bounded by the limits so computed, to include the population averages μ of the populations sampled. If in each instance we were to assert that μ is included within the limits computed, we should expect to be correct 95 times in 100 and in error 5 times in 100; that is, the statement “ μ is included within the interval so computed” has a probability of 0.95 of being correct. But there would be no operational meaning in the following statement made in any one instance: “The probability is 0.95 that μ falls within the limits computed in this case” since μ either does or does not fall within the limits. It should also be emphasized that even in repeated sampling from the *same* population, the interval defined by the limits $\bar{X} \pm as$ will vary in width and position from sample to sample, particularly with small samples (see Fig. 2). It is this series of ranges fluctuating in size and position that will include, ideally, the population mean μ 95 times out of 100 for $P = 0.95$.

These limits are commonly referred to as “confidence limits” [4,5]; for the three columns of Table 2, they may be referred to as the “90 % confidence limits,” “95 % confidence limits” and “99 % confidence limits,” respectively.

The magnitude $P = 0.95$ applies to the *series of samples* and is approached as a statistical limit as the number of instances in the series is increased indefinitely; hence, it signifies “statistical probability.” If the values of a given in Table 2 for $P = 0.99$ are used in a series of samples, we may, in like manner, expect 99 % of the sample intervals so computed to include the population mean μ .

Other values of P could, of course, be used if desired—the use of chances of 95 in 100, or 99 in 100 are, however, often found to be convenient in engineering presentations. Approximate values of a for other values of P may be read from the curves in Fig. 1 for samples of $n = 25$ or less.

For larger samples (n greater than 25), the constants 1.645, 1.960, and 2.576 in the expressions

$$a = \frac{1.645}{\sqrt{n}}, a = \frac{1.960}{\sqrt{n}}, \text{ and } a = \frac{2.576}{\sqrt{n}}$$

at the foot of Table 2 are obtained directly from normal law integral tables for probability values of 0.90, 0.95, and 0.99. To find the value of this constant for any other value of P , consult any standard text on statistical methods or read the value approximately on the “ k ” scale of Fig. 15 of **PART 1** of this Manual. For example, use of $a = 1/\sqrt{n}$ yields $P = 68.27\%$ and the limits ± 1 standard error, which some scientific journals print without noting a percentage.

2.5. EXPERIMENTAL ILLUSTRATION

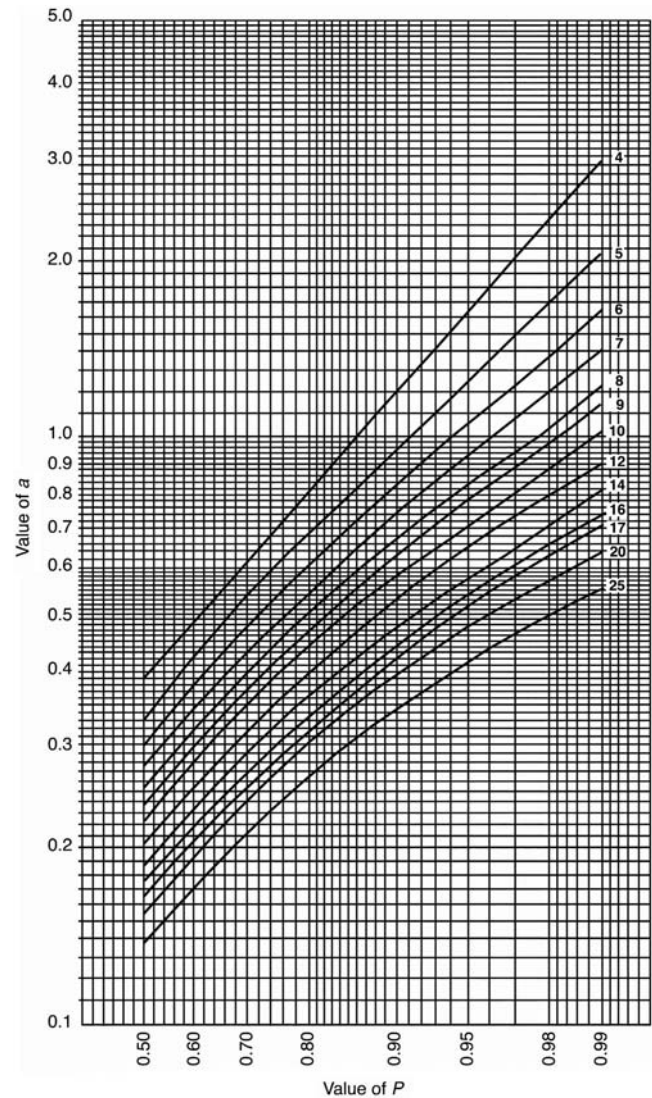
Figure 2 gives two diagrams illustrating the results of sampling experiments for samples of $n = 4$ observations each drawn from a normal population, for values of Case A, $P = 0.50$, and Case B, $P = 0.90$. For Case A, the intervals for 51 out of 100 samples included μ , and for Case B, 90 out of 100 included μ . If, in each instance (i.e., for each sample), we had concluded that the population mean μ is included within the limits shown for Case A, we would have been correct 51 times and in error 49 times, which is a

¹ If the population sampled is finite, that is, made up of a finite number of separate units that may be measured in respect to the variable, X , and if interest centers on the μ of this population, then this procedure assumes that the number of units, n , in the sample is relatively small compared with the number of units, N , in the population, say n is less than about 5 % of N . However, correction for relative size of sample can be made by multiplying s by the factor $\sqrt{1 - (n/N)}$. On the other hand, if interest centers on the μ of the underlying process or source of the finite population, then this correction factor is not used.

TABLE 2—Factors for Calculating 90, 95, and 99 % Confidence Limits for Averages^a

Number of Observations in Sample, n	Confidence Limits* $\bar{X} \pm as$		
	90 % Confidence Limits ($P = 0.90$) Value of a	95 % Confidence Limits ($P = 0.95$) Value of a	99 % Confidence Limits ($P = 0.99$) Value of a
4	1.177	1.591	2.921
5	0.953	1.241	2.059
6	0.823	1.050	1.646
7	0.734	0.925	1.401
8	0.670	0.836	1.237
9	0.620	0.769	1.118
10	0.580	0.715	1.028
11	0.546	0.672	0.955
12	0.518	0.635	0.897
13	0.494	0.604	0.847
14	0.473	0.577	0.805
15	0.455	0.554	0.769
16	0.438	0.533	0.737
17	0.423	0.514	0.708
18	0.410	0.497	0.683
19	0.398	0.482	0.660
20	0.387	0.468	0.640
21	0.376	0.455	0.621
22	0.367	0.443	0.604
23	0.358	0.432	0.588
24	0.350	0.422	0.573
25	0.342	0.413	0.559
n greater than 25	$a = \frac{1.645}{\sqrt{n}}$ approximately	$a = \frac{1.960}{\sqrt{n}}$ approximately	$a = \frac{2.576}{\sqrt{n}}$ approximately

* Limits that may be expected to include μ (9 times in 10, 95 times in 100, or 99 times in 100) in a series of problems, each involving a single sample of n observations.
Values of a are computed from Fisher, R.A., "Table of t ," *Statistical Methods for Research Workers*, Table IV based on Student's distribution of t .
^a Recomputed in 1975. The a of this table equals Fisher's t for $n - 1$ degree of freedom divided by \sqrt{n} . See also Fig. 1.

**FIG. 1—Curves giving factors for calculating 50 to 99 % confidence limits for averages (see also Table 2). Redrawn in 1975 for new values of a . Error in reading a not likely to be >0.01 . The numbers printed by the curves are the sample sizes (n).**

2.6. PRESENTATION OF DATA

In the presentation of data, if it is desired to give limits of this kind, it is quite important that the probability associated with the limits be clearly indicated. The three values $P = 0.90$, $P = 0.95$, and $P = 0.99$ given in Table 2 (chances of 9 in 10, 95 in 100, and 99 in 100) are arbitrary choices that may be found convenient in practice.

Example

Consider a sample of ten observations of breaking strength of hard-drawn copper wire as in Table 1, for which

$$\begin{aligned}\bar{X} &= 573.2 \text{ lb} \\ s &= 4.83 \text{ lb}\end{aligned}$$

Using this sample to define limits of uncertainty based on $P = 0.95$ (Table 2), we have

$$\begin{aligned}\bar{X} \pm 0.715s &= 573.2 \pm 3.5 \\ &= 569.7 \text{ and } 576.7\end{aligned}$$

reasonable variation from the expectancy of being correct 50 % of the time.

In this experiment, all samples were taken from the same population. However, the same reasoning applies to a series of samples that are each drawn from a population from the same universe as evidenced by conformance to the three conditions set forth in Section 2.4.

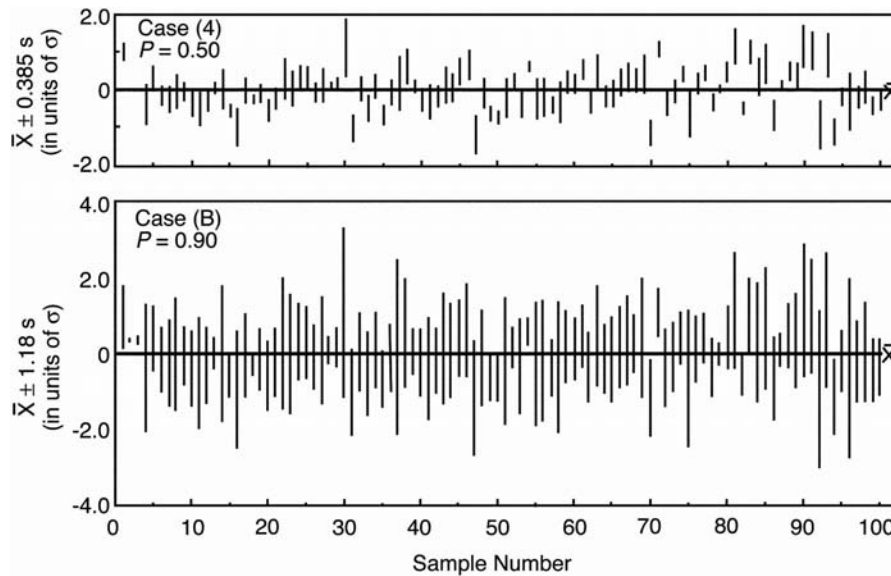


FIG. 2—Illustration showing computed intervals based on sampling experiments; 100 samples of $n = 4$ observations each, from a normal universe having $\mu = 0$ and $\sigma = 1$. Case A, are taken from Fig. 8 of Shewhart [2], and Case B gives corresponding intervals for limits $\bar{X} \pm 1.18s$, based on $P = 0.90$.

Two pieces of information are needed to supplement this numerical result: (a) the fact that 95 in 100 limits were used and (b) that this result is based solely on the evidence contained in ten observations. Hence, in the presentation of such limits, it is desirable to give the results in some way such as the following:

$$573.2 \pm 3.5 \text{ lb } (P = 0.95, n = 10)$$

The essential information contained in the data is, of course, covered by presenting \bar{X} , s , and n (see **PART 1** of this Manual), and the limits under discussion could be derived directly therefrom. If it is desired to present such limits in addition to \bar{X} , s , and n , the tabular arrangement given in Table 3 is suggested.

A satisfactory alternative is to give the plus or minus value in the column designated "Average, \bar{X} " and to add a note giving the significance of this entry, as shown in Table 4. If one omits the note, it will be assumed that $a = 1/\sqrt{n}$ was used and that $P = 68\%$.

2.7. ONE-SIDED LIMITS

Sometimes we are interested in limits of uncertainty in only one direction. In this case, we would present $\bar{X} + as$ or $\bar{X} - as$ (not both), a one-sided confidence limit below or above which the population mean may be expected to lie in a stated proportion of an indefinitely large number of samples. The a to use in this one-sided case and the associated confidence coefficient would be obtained from Table 2 or Fig. 1 as follows.

For a confidence coefficient of 0.95, use the a listed in Table 2 under $P = 0.90$.

For a confidence coefficient of 0.975, use the a listed in Table 2 under $P = 0.95$.

For a confidence coefficient of 0.995, use the a listed in Table 2 under $P = 0.99$.

In general, for a confidence coefficient of P_1 , use the a derived from Fig. 1 for $P = 1 - 2(1 - P_1)$. For example, with $n = 10$, $\bar{X} = 573.2$, and $s = 4.83$, the one-sided upper $P_1 = 0.95$ confidence limit would be to use $a = 0.58$ for $P = 0.90$ in Table 2, which yields $573.2 + 0.58(4.83) = 573.2 + 2.8 = 576.0$.

2.8. GENERAL COMMENTS ON THE USE OF CONFIDENCE LIMITS

In making use of limits of uncertainty of the type covered in this part, the engineer should keep in mind (1) the restrictions as to (a) controlled conditions, (b) approximate normality of population, and (c) randomness of sample and (2) the fact that the variability under consideration relates to fluctuations around the level of measurement values, whatever that may be, regardless of whether the population mean μ of the measurement values is widely displaced from the true value, μ_T , of what is being measured, as a result of the systematic or *constant* errors present throughout the measurements.

For example, breaking strength values might center around a value of 575.0 lb (the population mean μ of the measurement values) with a scatter of individual observations represented by the dotted distribution curve of Fig. 3, whereas the

TABLE 3—Suggested Tabular Arrangement

Number of Tests, n	Average, \bar{X}	Limits for μ (95 % Confidence Limits)	Standard Deviation, s
10	573.2	573.2 ± 3.5	4.83

TABLE 4—Alternative to Table 3

Number of Tests, n	Average, \bar{X}^a	Standard Deviation, s
10	$573.2 (\pm 3.5)$	4.83
^a The \pm entry indicates 95 % confidence limits for μ .		

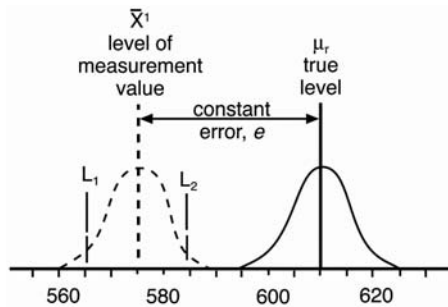


FIG. 3—Plot shows how plus or minus limits (L_1 and L_2) are unrelated to a systematic or constant error.

true average μ_T for the batch of wire under test is actually 610.0 lb, the difference between 575.0 and 610.0 representing a constant or systematic error present in *all* the observations as a result, say, of an incorrect adjustment of the testing machine.

The limits thus have meaning for series of like measurements, made under like conditions, *including* the same constant errors if any be present.

In the practical use of these limits, the engineer may not have assurance that conditions (a), (b), and (c) given in Section 2.4 are met; hence, it is not advisable to place great emphasis on the exact magnitudes of the probabilities given in Table 2 but rather to consider them as orders of magnitude to be used as general guides.

2.9. NUMBER OF PLACES TO BE RETAINED IN COMPUTATION AND PRESENTATION

The following working rule is recommended in carrying out computations incident to determining averages, standard deviations, and “limits for averages” of the kind here considered, for a sample of n observed values of a variable quantity:

In all operations on the sample of n observed values, such as adding, subtracting, multiplying, dividing, squaring, extracting square root, etc., retain the equivalent of two more places of figures than in the single observed values. For example, if observed values are read or determined to the nearest 1 lb, carry numbers to the nearest 0.01 lb in the computations; if observed values are read or determined to the nearest 10 lb, carry numbers to the nearest 0.1 lb in the computations, etc.

Deleting places of figures should be done after computations are completed, in order to keep the final results substantially free from computation errors. In deleting places of figures, the actual rounding-off procedure should be carried out as follows.²

1. When the figure next beyond the last figure or place to be retained is less than 5, the figure in the last place retained should be kept unchanged.
2. When the figure next beyond the last figure or place to be retained is more than 5, the figure in the last place retained should be increased by 1.
3. When the figure next beyond the last figure or place to be retained is 5, and (a) there are no figures, or only zeros, beyond this 5, if the figure in the last place to be retained is odd, it should be increased by 1; if even, it should be kept unchanged; but (b) if the 5 next beyond the figure in the last place to be retained is followed by any figures other than zero, the figure in the last place retained should be increased by 1, whether odd or even. For example, if in the following numbers, the places of figures in parentheses are to be rejected,

39,4(49) becomes 39,400,
 39,4(50) becomes 39,400,
 39,4(51) becomes 39,500, and
 39,5(50) becomes 39,600.

The number of places of figures to be retained in the presentation depends on what use is to be made of the results and on the sampling variation present. No general rule, therefore, can safely be laid down. The following working rule has, however, been found generally satisfactory by ASTM E11.30 Subcommittee on Statistical Quality Control in presenting the results of testing in technical investigations and development work:

- a. See Table 5 for averages.
- b. For standard deviations, retain three places of figures.
- c. If “limits for averages” of the kind here considered are presented, retain the same places of figures as are retained for the average.

For example, if $n = 10$, and if observed values were obtained to the nearest 1 lb, present averages and “limits for averages” to the nearest 0.1 lb, and present the standard deviation to three places of figures. This is illustrated in the tabular presentation in Section 2.6.

TABLE 5—Averages			
When the Single Values Are Obtained to the Nearest	And When the Number of Observed Values Is		
0.1, 1, 10, etc., units	...	2 to 20	21 to 200
0.2, 2, 20, etc., units	less than 4	4 to 40	41 to 400
0.5, 5, 50, etc., units	less than 10	10 to 100	101 to 1,000
Retain the following number of places of figures in the average	{ same number of places as in single values	1 more place than in single values	2 more places than in single values

² This rounding-off procedure agrees with that adopted in ASTM Recommended Practice for Using Significant Digits in Test Data to Determine Conformance with Specifications (E29).

TABLE 6—Effect of Rounding							
	Not Rounded				Rounded		
	Limits		Difference		Limits		Difference
573.5 ± 1.4	572.1	574.9	2.8	574 ± 1	573	575	2
573.5 ± 1.5	572.0	575.0	3.0	574 ± 2	572	576	4

Rule (a) will result generally in one and conceivably in two doubtful places of figures in the average—that is, places that may have been affected by the rounding-off (or observation) of the n individual values to the nearest number of units stated in the first column of the table. Referring to Tables 3 and Table 4, the third place figures in the average, $\bar{X} = 573.2$, corresponding to the first place of figures in the ± 3.5 value, are doubtful in this sense. One might conclude that it would be suitable to present the average to the nearest pound, thus,

$$573 \pm 3 \text{ lb} (P = 0.95, n = 10)$$

This might be satisfactory for some purposes. However, the effect of such rounding-off to the first place of figures of the plus or minus value may be quite pronounced if the first digit of the plus or minus value is small, as indicated in Table 6. If further use were to be made of these data—such as collecting additional observations to be combined with these, gathering other data to be compared with these, etc.—then the effect of such rounding-off of \bar{X} in a presentation might seriously interfere with proper subsequent use of the information.

The number of places of figures to be retained or to be used as a basis for action in specific cases cannot readily be made subject to any general rule. It is, therefore, recommended that in such cases, the number of places be settled by definite agreements between the individuals or parties involved. In reports covering the acceptance and rejection of material, ASTM E29, Standard Practice for Using Significant Digits in Test Data to Determine Conformance with Specifications, gives specific rules that are applicable when reference is made to this recommended practice.

SUPPLEMENT 2.A

Presenting Plus or Minus Limits of Uncertainty for σ —Normal Distribution³

When observations X_1, X_2, \dots, X_n are made under controlled conditions, and there is reason to believe the distribution of X is normal, two-sided confidence limits for the standard deviation of the population with confidence coefficient, P , will be given by the lower confidence limit for

$$\hat{\sigma}_L = s \sqrt{(n-1)/\chi_{(1-P)/2}^2} \quad (1)$$

And the upper confidence limit for

$$\hat{\sigma}_U = s \sqrt{(n-1)/\chi_{(1+P)/2}^2}$$

where the quantity $\chi_{(1-P)/2}^2$ (or $\chi_{(1+P)/2}^2$) is the χ^2 value of a chi-square variable with $n-1$ degrees of freedom, which is exceeded with probability $(1-P)/2$ or $(1+P)/2$ as found in most statistics textbooks.

To facilitate computation, Table 7 gives values of

$$b_L = \sqrt{(n-1)/\chi_{(1-P)/2}^2} \quad \text{and} \quad (2)$$

$$b_U = \sqrt{(n-1)/\chi_{(1+P)/2}^2}$$

for $P = 0.90, 0.95$, and 0.99 . Thus we have, for a normal distribution, the estimate of the lower confidence limit for σ as

$$\hat{\sigma}_L = b_L s$$

and for the upper confidence limit

$$\hat{\sigma}_U = b_U s \quad (3)$$

Example

Table 1 of **PART 2** gives the standard deviation of a sample of ten observations of breaking strength of copper wire as $s = 4.83$ lb. If we assume that the breaking strength has a normal distribution, which may actually be somewhat questionable, we have as 0.95 confidence limits for the universe standard deviation σ that yield a lower 0.95 confidence limit of

$$\hat{\sigma}_L = 0.688(4.83) = 3.32 \text{ lb}$$

and an upper 0.95 confidence limit of

$$\hat{\sigma}_U = 1.83(4.83) = 8.83 \text{ lb}$$

If we wish a one-sided confidence limit on the low side with confidence coefficient P , we estimate the lower one-sided confidence limit as

$$\hat{\sigma}_L = s \sqrt{(n-1)/\chi_{(1-P)}^2}$$

For a one-sided confidence limit on the high side with confidence coefficient P , we estimate the upper one-sided confidence limit as

$$\hat{\sigma}_U = s \sqrt{(n-1)/\chi_P^2}$$

Thus, for $P = 0.95, 0.975$, and 0.995 , we use the b_L or b_U factor from Table 7 in the columns headed 0.90, 0.95, and 0.99, respectively. For example, a 0.95 upper one-sided

³ The analysis is strictly valid only for an unlimited population such as presented by a manufacturing or measurement process. When the population sampled is relatively small compared with the sample size n , the reader is advised to consult a statistician.

confidence limit for σ based on a sample of ten items for which $s = 4.83$ would be

$$\begin{aligned}\hat{\sigma}_U &= b_{U(0.90)}s \\ &= 1.64(4.83) \\ &= 7.92\end{aligned}$$

A lower 0.95 one-sided confidence limit would be

$$\begin{aligned}\hat{\sigma}_L &= b_{L(0.90)}s \\ &= 0.730(4.83) \\ &= 3.53\end{aligned}$$

TABLE 7— b -Factors for Calculating Confidence Limits for σ , Normal Distribution^a

Number of Observations in Sample, n	90 % Confidence Limits		95 % Confidence Limits		99 % Confidence Limits	
	b_L	b_U	b_L	b_U	b_L	b_U
2	0.510	16.0	0.446	31.9	0.356	159.5
3	0.578	4.41	0.521	6.29	0.434	14.1
4	0.619	2.92	0.566	3.73	0.484	6.47
5	0.649	2.37	0.600	2.87	0.518	4.40
6	0.671	2.09	0.625	2.45	0.547	3.48
7	0.690	1.91	0.645	2.20	0.569	2.98
8	0.705	1.80	0.661	2.04	0.587	2.66
9	0.718	1.71	0.676	1.92	0.603	2.44
10	0.730	1.64	0.688	1.83	0.618	2.28
11	0.739	1.59	0.698	1.75	0.630	2.15
12	0.747	1.55	0.709	1.70	0.641	2.06
13	0.756	1.51	0.718	1.65	0.651	1.98
14	0.762	1.49	0.725	1.61	0.660	1.91
15	0.769	1.46	0.732	1.58	0.669	1.85
16	0.775	1.44	0.739	1.55	0.676	1.81
17	0.780	1.42	0.745	1.52	0.683	1.76
18	0.785	1.40	0.750	1.50	0.690	1.73
19	0.789	1.38	0.756	1.48	0.696	1.70
20	0.794	1.37	0.760	1.46	0.702	1.67
21	0.798	1.35	0.765	1.44	0.707	1.64
22	0.801	1.35	0.769	1.43	0.712	1.62
23	0.806	1.34	0.773	1.41	0.717	1.60
24	0.808	1.33	0.777	1.40	0.721	1.58
25	0.812	1.32	0.780	1.39	0.725	1.56
26	0.814	1.31	0.785	1.38	0.730	1.54
27	0.818	1.30	0.788	1.37	0.734	1.52
28	0.821	1.29	0.791	1.36	0.738	1.51
29	0.823	1.29	0.793	1.35	0.741	1.50
30	0.825	1.28	0.797	1.35	0.745	1.49
31	0.828	1.27	0.799	1.34	0.747	1.47
For larger n and	$1/(1 + 1.645/\sqrt{2n})$ $1/(1 - 1.645/\sqrt{2n})$		$1/(1 + 1.960/\sqrt{2n})$ $1/(1 - 1.960/\sqrt{2n})$		$1/(1 + 2.576/\sqrt{2n})$ $1/(1 - 2.576/\sqrt{2n})$	

^a Confidence limits for $\sigma = b_L s$ and $b_U s$.

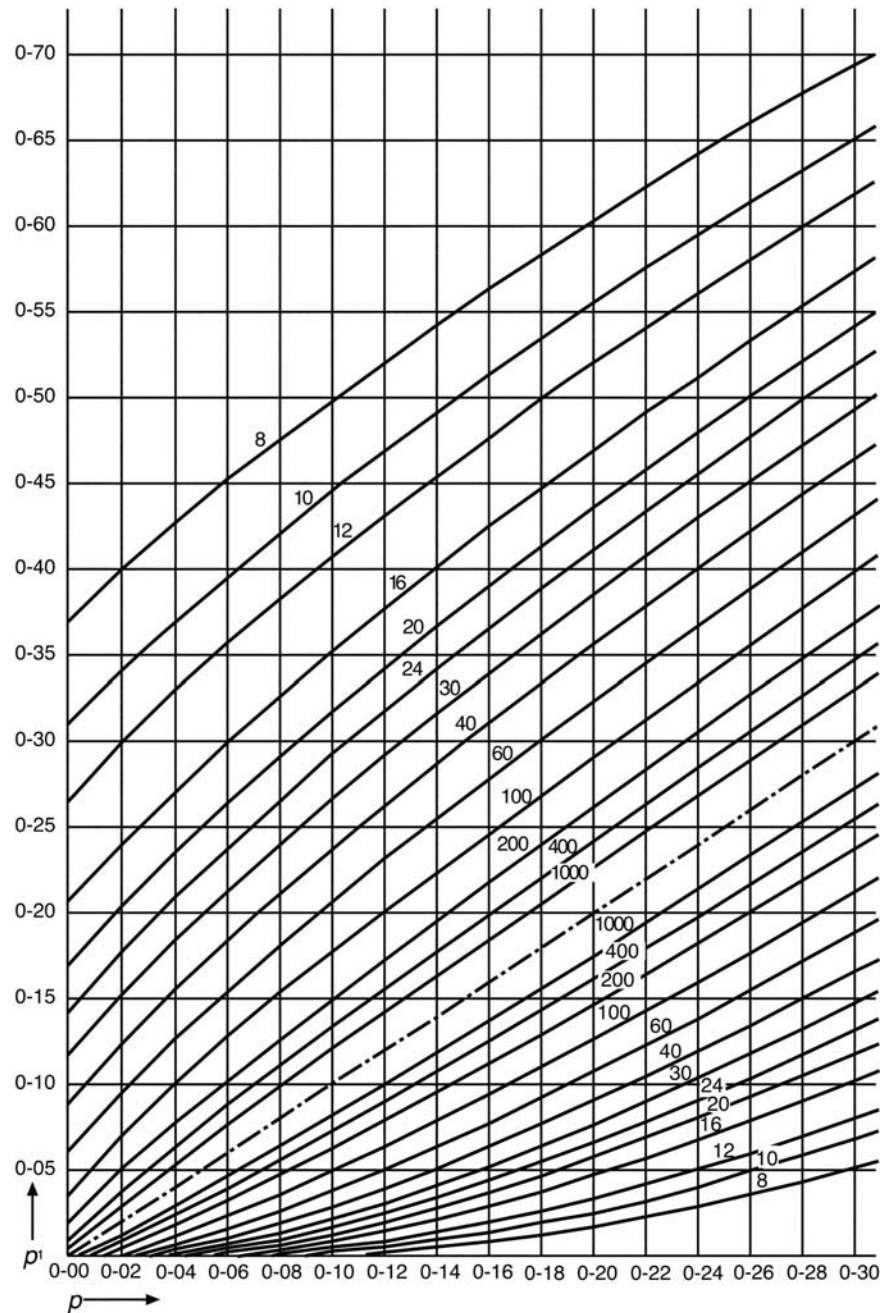


FIG. 4—Chart providing confidence limits for p' in binomial sampling, given a sample fraction. Confidence coefficient = 0.95. The numbers printed along the curves indicate the sample size n . If for a given value of the abscissa, p_A and p_B are the ordinates read from (or interpolated between) the appropriate lower and upper curves, then $Pr\{p_A \leq p' \leq p_B\} \geq 0.95$. Reproduced by permission of the *Biometrika Trust*.

SUPPLEMENT 2.B

Presenting Plus or Minus Limits of Uncertainty for p^4

When there is a fraction p of a given category, for example, the fraction nonconforming, in n observations obtained under controlled conditions, 95 % confidence limits for the population fraction p' may be found in the chart in Fig. 41 of *Biometrika Tables for Statisticians*, Vol. 1. A reproduction of this fraction is entered on the abscissa and the upper and lower 0.95 confidence limits are charted for selected sample

sizes and shown in Fig. 4. To use the chart, the sample fraction is entered on the abscissa and the upper and lower 0.95 confidence limits are read on the vertical scale for various values of n . Approximate limits for values of n not shown on the *Biometrika* chart may be obtained by graphical interpolation. The *Biometrika Tables for Statisticians* also give a chart for 0.99 confidence limits.

In general, for an np and $n(1 - p)$ of at least 6 and preferably $0.10 \leq p \leq 0.90$, the following formulas can be applied

⁴ The analysis is strictly valid only for an unlimited population such as presented by a manufacturing or measurement process. When the population sampled is relatively small compared with the sample size n , the reader is advised to consult a statistician.

approximate 0.90 confidence limits

$$p \pm 1.645\sqrt{p(1-p)/n}$$

approximate 0.95 confidence limits

$$p \pm 1.960\sqrt{p(1-p)/n} \quad (4)$$

approximate 0.99 confidence limits

$$p \pm 2.576\sqrt{p(1-p)/n}.$$

Example

Refer to the data of Table 2(a) of **PART 1** and Fig. 4 of **PART 1** and suppose that the lower specification limit on transverse strength is 675 psi and there is no upper specification limit. Then, the sample percentage of bricks nonconforming (the sample “fraction nonconforming” p) is seen to be $8/270 = 0.030$. Rough 0.95 confidence limits for the universe fraction nonconforming p' are read from Fig. 4 as 0.02 to 0.07. Using Eq (4), we have approximate 95 % confidence limits as

$$\begin{aligned} 0.030 \pm 1.960\sqrt{0.030(1-0.030/270)} \\ = 0.030 \pm 1.960(0.010) \\ = \begin{cases} 0.05 \\ 0.01 \end{cases} \end{aligned}$$

Even though $p > 0.10$, the two results agree reasonably well.

One-sided confidence limits for a population fraction p' can be obtained directly from the Biometrika chart or Fig. 4,

but the confidence coefficient will be 0.975 instead of 0.95 as in the two-sided case. For example, with $n = 200$ and the sample $p = 0.10$, the 0.975 upper one-sided confidence limit is read from Fig. 4 to be 0.15. When the Normal approximation can be used, we will have the following approximate one-sided confidence limits for p' .

$$\begin{aligned} P = 0.90 : & \text{lowerlimit} = p - 1.282\sqrt{p(1-p)/n} \\ & \text{upperlimit} = p + 1.282\sqrt{p(1-p)/n} \\ P = 0.95 : & \text{lowerlimit} = p - 1.645\sqrt{p(1-p)/n} \\ & \text{upperlimit} = p + 1.645\sqrt{p(1-p)/n} \\ P = 0.99 : & \text{lowerlimit} = p - 2.326\sqrt{p(1-p)/n} \\ & \text{upperlimit} = p + 2.326\sqrt{p(1-p)/n} \end{aligned}$$

References

- [1] Shewhart, W.A., “Probability as a Basis for Action,” presented at the joint meeting of the American Mathematical Society and Section K of the A.A.A.S., 27 Dec. 1932.
- [2] Shewhart, W.A., Statistical Method from the Viewpoint of Quality Control, W. E. Deming, Ed., The Graduate School, Department of Agriculture, Washington, D.C., 1939.
- [3] Pearson, E.S., The Application of Statistical Methods to Industrial Standardization and Quality Control, B.S. 600-1935, British Standards Institution, London, Nov. 1935.
- [4] Snedecor, G.W. and Cochran, W.G., Statistical Methods, 7th ed., Iowa State University Press, Ames, IA, 1980, pp. 54–56.
- [5] Ott, E.R., Schilling, E.G., and Neubauer, D.V., Process Quality Control, 4th ed., McGraw-Hill, New York, NY, 2005.

Control Chart Method of Analysis and Presentation of Data

GLOSSARY OF TERMS AND SYMBOLS USED IN PART 3

In general, the terms and symbols used in **PART 3** have the same meanings as in preceding parts of the Manual. In a few cases, which are indicated in the following glossary, a more specific meaning is attached to them for the convenience of a portion or all of **PART 3**. Mathematical definitions and derivations are given in Supplement 3.A.

GLOSSARY OF TERMS

assignable cause, n.—identifiable factor that contributes to variation in quality and which it is feasible to detect and identify. Sometimes referred to as a special cause.

chance cause, n.—identifiable factor that exhibits variation that is random and free from any recognizable pattern over time. Sometimes referred to as a common cause.

lot, n.—definite quantity of some commodity produced under conditions that are considered uniform for sampling purposes.

sample, n.—group of units, or portion of material, taken from a larger collection of units or quantity of material, which serves to provide information that can be used as a basis for action on the larger quantity or on the production process. May be referred to as a subgroup in the construction of a control chart.

subgroup, n.—one of a series of groups of observations obtained by subdividing a larger group of observations; alternatively, the data obtained from one of a series of samples taken from a series of lots or from sublots taken from a process. One of the essential features of the control chart method is to break up the inspection data into rational subgroups, that is, to classify the observed values into subgroups, within which variations may, for engineering reasons, be considered to be due to nonassignable chance causes only but between which there may be differences due to one or more assignable causes whose presence is considered possible. May be

Glossary of Symbols

Symbol	General	In PART 3 , Control Charts
c		<i>number of nonconformities</i> ; more specifically the number of nonconformities in a sample (subgroup)
c_4		factor that is a function of n and expresses the ratio between the expected value of \bar{s} for a large number of samples of n observed values each and the σ of the universe sampled. (Values of $c_4 = E(\bar{s})/\sigma$ are given in Tables 6 and 16, and in Table 49 in Supplement 3.A, based on a normal distribution.)
d_2		factor that is a function of n and expresses the ratio between expected value of \bar{R} for a large number of samples of n observed values each and the σ of the universe sampled. (Values of $d_2 = E(\bar{R})/\sigma$ are given in Tables 6 and 16, and in Table 49 in Supplement 3.A, based on a normal distribution.)
k		number of subgroups or samples under consideration
MR	typically the absolute value of the difference of two successive values plotted on a control chart. It may also be the range of a group of more than two successive values.	absolute value of the difference of two successive values plotted on a control chart.
\overline{MR}	average of $n - 1$ moving ranges from a series of n values	average moving range of $n - 1$ moving ranges from a series of n values $\overline{MR} = \frac{ X_2 - X_1 + \dots + X_n - X_{n-1} }{n-1}$

n	number of observed values (observations)	subgroup or sample size, that is, the number of units or observed values in a sample or subgroup
p	relative frequency or proportion, the ratio of the number of occurrences to the total possible number of occurrences	fraction nonconforming, the ratio of the number of nonconforming units (articles, parts, specimens, etc.) to the total number of units under consideration; more specifically, the fraction nonconforming of a sample (subgroup)
np	number of occurrences	number of nonconforming units; more specifically, the number of nonconforming units in a sample of n units
R	range of a set of numbers, that is, the difference between the largest number and the smallest number	range of the n observed values in a subgroup (sample) (the symbol R is also used to designate the moving range in 29 and 30)
s	sample standard deviation	standard deviation of the n observed values in a subgroup (sample) $s = \sqrt{\frac{(X_1 - \bar{X})^2 + \cdots + (X_n - \bar{X})^2}{n - 1}}$ or expressed in a form more convenient but sometimes less accurate for computation purposes $s = \sqrt{\frac{n(X_1^2 + \cdots + X_n^2) - (X_1 + \cdots + X_n)^2}{n(n - 1)}}$
u		nonconformities per units, the number of nonconformities in a sample of n units divided by n
X	observed value of a measurable characteristic; specific observed values are designated X_1, X_2, X_3 , etc.; also used to designate a measurable characteristic	
\bar{X}	average (arithmetic mean); the sum of the n observed values divided by n	average of the n observed values in a subgroup (sample) $\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$
Qualified Symbols		
$\sigma_{\bar{X}}, \sigma_s, \sigma_R, \sigma_p$; etc.	standard deviation of values of \bar{X}, s, R, p , etc.	standard deviation of the sampling distribution of \bar{X}, s, R, p , etc.
$\bar{\bar{X}}, \bar{s}, \bar{R}, \bar{p}$, etc.	average of a set of values of \bar{X}, s, R, p , etc. (the over-bar notation signifies an average value)	average of the set of k subgroup (sample) values of \bar{X}, s, R, p , etc., under consideration; for samples of unequal size, an overall or weighted average
μ, σ, p', u', c'	mean, standard deviation, fraction nonconforming, etc., of the population	
$\mu_0, \sigma_0, p_0, u_0, c_0$		standard value of μ, σ, p' , etc., adopted for computing control limits of a control chart for the case, Control with Respect to a Given Standard (see Sections 3.18 to 3.27)
α	alpha risk of claiming that a hypothesis is true when it is actually true	risk of claiming that a process is out of statistical control when it is actually in statistical control, a.k.a., Type I error. $100(1 - \alpha)$ % is the percent confidence.
β	beta risk of claiming that a hypothesis is false when it is actually false	risk of claiming that a process is in statistical control when it is actually out of statistical control, a.k.a., Type II error. $100(1 - \beta)$ % is the power of a test that declares the hypothesis is false when it is actually false.

referred to as a sample from the process in the construction of a control chart.

unit, n.—one of a number of similar articles, parts, specimens, lengths, areas, etc. of a material or product.

sublot, n.—identifiable part of a lot.

GENERAL PRINCIPLES

3.1. PURPOSE

PART 3 of the Manual gives formulas, tables, and examples that are useful in applying the *control chart* method [1] of

analysis and presentation of data. This method requires that the data be obtained from several samples or that the data be capable of subdivision into subgroups based on relevant engineering information. Although the principles of **PART 3** are applicable generally to many kinds of data, they will be discussed herein largely in terms of the quality of materials and manufactured products.

The control chart method provides a criterion for detecting lack of statistical control. Lack of statistical control in data indicates that observed variations in quality are greater than should be attributed to chance. Freedom from indications of lack of control is necessary for scientific evaluation of data and the determination of quality.

The control chart method lays emphasis on the *order* or grouping of the observations in a set of individual observations, sample averages, number of nonconformities, etc., with respect to time, place, source, or any other consideration that provides a basis for a classification that may be of significance in terms of known conditions under which the observations were obtained.

This concept of order is illustrated by the data in Table 1 in which the width in inches to the nearest 0.0001-in. is given for 60 specimens of Grade BB zinc that were used in ASTM atmospheric corrosion tests.

At the left of the table, the data are tabulated without regard to relevant information. At the right, they are shown arranged in ten subgroups, where each subgroup relates to the specimens from a separate milling. The information regarding origin is relevant engineering information, which makes it possible to apply the control chart method to these data (see *Example 3*).

3.2. TERMINOLOGY AND TECHNICAL BACKGROUND

Variation in quality from one unit of product to another is usually due to a very large number of causes. Those causes, for which it is possible to identify, are termed *special causes*, or *assignable causes*. Lack of control indicates one or more assignable causes are operative. The vast majority of causes of variation may be found to be inconsequential and cannot be identified. These are termed *chance causes*, or *common*

TABLE 1—Comparison of Data Before and After Subgrouping (Width in Inches of Specimens of Grade BB zinc)

Before Subgrouping			After Subgrouping						
			Specimen						
0.5005	0.5005	0.4996	Subgroup (Milling)						
0.5000	0.5002	0.4997		1	2	3	4	5	6
0.5008	0.5003	0.4993							
0.5000	0.5004	0.4994	1	0.5005	0.5000	0.5008	0.5000	0.5005	0.5000
0.5005	0.5000	0.4999							
0.5000	0.5005	0.4996	2	0.4998	0.4997	0.4998	0.4994	0.4999	0.4998
0.4998	0.5008	0.4996							
0.4997	0.5007	0.4997	3	0.4995	0.4995	0.4995	0.4995	0.4995	0.4996
0.4998	0.5008	0.4995							
0.4994	0.5010	0.4995	4	0.4998	0.5005	0.5005	0.5002	0.5003	0.5004
0.4999	0.5008	0.4997							
0.4998	0.5009	0.4992	5	0.5000	0.5005	0.5008	0.5007	0.5008	0.5010
0.4995	0.5010	0.4995							
0.4995	0.5005	0.4992	6	0.5008	0.5009	0.5010	0.5005	0.5006	0.5009
0.4995	0.5006	0.4994							
0.4995	0.5009	0.4998	7	0.5000	0.5001	0.5002	0.4995	0.4996	0.4997
0.4995	0.5000	0.5000							
0.4996	0.5001	0.4990	8	0.4993	0.4994	0.4999	0.4996	0.4996	0.4997
0.4998	0.5002	0.5000							
0.5005	0.4995	0.5000	9	0.4995	0.4995	0.4997	0.4995	0.4995	0.4992
			10	0.4994	0.4998	0.5000	0.4990	0.5000	0.5000

causes. However, causes of large variations in quality generally admit of ready identification.

In more detail we may say that for a constant system of chance causes, the average \bar{X} , the standard deviations s , the value of fraction nonconforming p , or any other functions of the observations of a series of samples will exhibit statistical stability of the kind that may be expected in random samples from homogeneous material. The criterion of the quality control chart is derived from laws of chance variations for such samples, and failure to satisfy this criterion is taken as evidence of the presence of an operative assignable cause of variation.

As applied by the manufacturer to inspection data, the control chart provides a basis for *action*. Continued use of the control chart and the elimination of assignable causes as their presence is disclosed, by failures to meet its criteria, tend to reduce variability and to stabilize quality at aimed-at levels [2–9]. While the control chart method has been devised primarily for this purpose, it provides simple techniques and criteria that have been found useful in analyzing and interpreting other types of data as well.

3.3. TWO USES

The control chart method of analysis is used for the following two distinct purposes.

A. Control—No Standard Given

To discover whether observed values of \bar{X} , s , p , etc., for several samples of n observations each, *vary among themselves* by an amount greater than should be attributed to chance. Control charts based entirely on the data from samples are used for detecting *lack of constancy* of the cause system.

B. Control with Respect to a Given Standard

To discover whether observed values of \bar{X} , s , p , etc., for samples of n observations each, differ from standard values μ_0 , σ_0 , p_0 , etc., by an amount greater than should be attributed to chance. The standard value may be an experience value based on representative prior data, or an economic value established on consideration of needs of service and cost of production, or a desired or aimed-at value designated by specification. It should be noted particularly that the standard value of σ , which is used not only for setting up control charts for s or R but also for computing control limits on control charts for \bar{X} , should almost invariably be an experience value based on representative *prior* data. Control charts based on such standards are used particularly in inspection to control processes and to maintain quality uniformly at the level desired.

3.4. BREAKING UP DATA INTO RATIONAL SUBGROUPS

One of the essential features of the control chart method is what is referred to as breaking up the data into rationally chosen subgroups called *rational subgroups*. This means classifying the observations under consideration into subgroups or samples, *within* which the variations may be considered on engineering grounds to be due to nonassignable chance causes only, but *between* which the differences may be due to assignable causes whose presence are suspected or considered possible.

This part of the problem depends on technical knowledge and familiarity with the conditions under which the material sampled was produced and the conditions under which the data were taken. By identifying each sample with a time or a source, specific causes of trouble may be more readily traced and corrected, if advantageous and economical. Inspection and test records, giving observations in the order in which they were taken, provide directly a basis for subgrouping with respect to time. This is commonly advantageous in manufacture where it is important, from the standpoint of quality, to maintain the production cause system constant with time.

It should always be remembered that analysis will be greatly facilitated if, when planning for the collection of data in the first place, care is taken to so select the samples that the data from each sample can properly be treated as a separate rational subgroup, and that the samples are identified in such a way as to make this possible.

3.5. GENERAL TECHNIQUE IN USING CONTROL CHART METHOD

The general technique (see Ref. 1, Criterion I, Chapter XX) of the control chart method variations in quality generally admit of ready identification is as follows. Given a set of observations, to determine whether an assignable cause of variation is present:

- Classify the total number of observations into k rational subgroups (samples) having n_1, n_2, \dots, n_k observations, respectively. Make subgroups of *equal size*, if practicable. It is usually preferable to make subgroups not smaller than $n = 4$ for variables \bar{X} , s , or R , nor smaller than $n = 25$ for (binary) attributes. (See Sections 3.13, 3.15, 3.23, and 3.25 for further discussion of preferred sample sizes and subgroup expectancies for general attributes.)
- For each statistic (\bar{X} , s , R , p , etc.) to be used, construct a control chart with control limits in the manner indicated in the subsequent sections.
- If one or more of the observed values of \bar{X} , s , R , p , etc., for the k subgroups (samples) fall outside the control limits, take this fact as an indication of the presence of an assignable cause.

3.6. CONTROL LIMITS AND CRITERIA OF CONTROL

In both uses indicated in Section 3.3, the control chart consists essentially of symmetrical limits (control limits) placed above and below a central line. The central line in each case indicates the expected or average value of \bar{X} , s , R , p , etc. for subgroups (samples) of n observations each.

The control limits used here, referred to as *3-sigma control limits*, are placed at a distance of three standard deviations from the central line. The standard deviation is defined as the standard deviation of the sampling distribution of the statistical measure in question (\bar{X} , s , R , p , etc.) for subgroups (samples) of size n . Note that this standard deviation is *not* the standard deviation computed from the subgroup values (of \bar{X} , s , R , p , etc.) plotted on the chart but is computed from the variations within the subgroups (see Supplement 3.B, *Note 1*).

Throughout this part of the Manual such standard deviations of the sampling distributions will be designated as $\sigma_{\bar{X}}$, σ_s , σ_R , σ_p , etc., and these symbols, which consist of σ and a subscript, will be used only in this restricted sense.

For measurement data, if μ and σ were known, we would have

Control limits for:	
average	(expected \bar{X}) $\pm 3\sigma_{\bar{X}}$
standard deviations	(expected s) $\pm 3\sigma_s$
ranges	(expected R) $\pm 3\sigma_R$

where the various expected values are derived from estimates of μ or σ . For attribute data, if p' were known, we would have control limits for values of p (expected p) $\pm 3\sigma_p$, where expected $p = p'$.

The use of 3-sigma control limits can be attributed to Walter Shewhart, who based this practice upon the evaluation of numerous datasets [1]. Shewhart determined that based on a single point relative to 3-sigma control limits the control chart would signal assignable causes affecting the process. Use of 4-sigma control limits would not be sensitive enough, and use of 2-sigma control limits would produce too many false signals (too sensitive) based on the evaluation of a single point.

Figure 1 indicates the features of a control chart for averages. The choice of the factor 3 (a multiple of the expected standard deviation of \bar{X} , s , R , p , etc.) in these limits, as Shewhart suggested [1], is an economic choice based on experience that covers a wide range of industrial applications of the control chart, rather than on any exact value of probability (see Supplement 3.B, Note 2). This choice has proved satisfactory for use as a criterion for *action*, that is, for looking for assignable causes of variation.

This *action* is presumed to occur in the normal work setting, where the cost of too frequent false alarms would be uneconomic. Furthermore, the situation of too frequent false alarms could lead to a rejection of the control chart as a tool if such deviations on the chart are of no practical or engineering significance. In such a case, the control limits

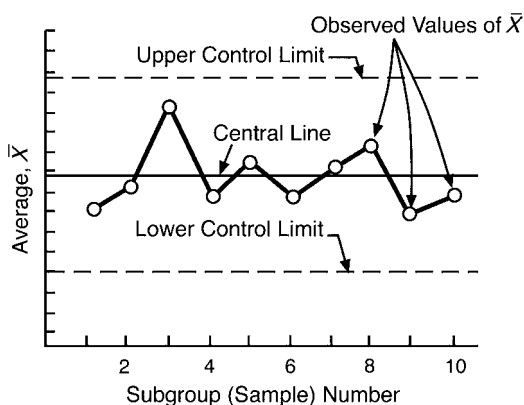


FIG. 1—Essential features of a control chart presentation; chart for averages.

should be reevaluated to determine if they correctly reflect the system of chance, or common, cause variation of the process. For example, a control chart on a raw material assay may have understated control limits if the data on which they were based encompassed only a single lot of the raw material. Some lot-to-lot raw material variation would be expected since nature is in control of the assay of the material as it is being mined. Of course, in some cases, some compensation by the supplier may be possible to correct problems with particle size and the chemical composition of the material in order to comply with the customer's specification.

In *exploratory* research, or in the early phases of a deliberate investigation into potential improvements, it may be worthwhile to investigate points that fall outside what some have called a set of *warning limits* (often placed two standard deviations about the centerline). The chances that any single point would fall two standard deviations from the average is roughly 1:20, or 5% of the time, when the process is indeed centered and in statistical control. Thus, stopping to investigate a false alarm once for every 20 plotting points on a control chart would be too excessive. Alternatively, an effective *rule of nonrandomness* would be to take action if two consecutive points were beyond the warning limits on the same side of the centerline. The *risk* of such an action would only be roughly 1:800! Such an occurrence would be considered an unlikely event and indicate that the process is not in control, so justifiable action would be taken to identify an assignable cause.

A control chart may be said to display a lack of control under a variety of circumstances, any of which provide some evidence of nonrandom behavior. Several of the best known nonrandom patterns can be detected by the manner in which one or more tests for nonrandomness are violated. The following list of such tests are given below:

1. Any single point beyond 3σ limits
2. Two consecutive points beyond 2σ limits on the same side of the centerline
3. Eight points in a row on one side of the centerline
4. Six points in a row that are moving away or toward the centerline with no change in direction (a.k.a., trend rule)
5. Fourteen consecutive points alternating up and down (sawtooth pattern)
6. Two of three points beyond 2σ limits on the same side of the centerline
7. Four of five points beyond 1σ limits on the same side of the centerline
8. Fifteen points in a row within the 1σ limits on either side of the centerline (a.k.a., stratification rule—sampling from two sources *within* a subgroup)
9. Eight consecutive points outside the 1σ limits on both sides of the centerline (a.k.a., mixture rule—sampling from two sources *between* subgroups)

There are other rules that can be applied to a control chart in order to detect nonrandomness, but those given here are the most common rules in practice.

It is also important to understand what risks are involved when implementing control charts on a process. If we state that *the process is in a state of statistical control*, and present it as a hypothesis, then we can consider what risks are operative in any process investigation. In particular,

there are two types of risk that can be seen in the following table:

Decision about the State of the Process Based on Data	True State of the Process	
	Process is IN Control	Process is OUT of Control
Process is IN control	No error is made	Beta (β) risk, or Type II error
Process is OUT of control	Alpha (α) risk, or Type I error	No error is made

For a set of data analyzed by the control chart method, when may a state of control be assumed to exist? Assuming subgrouping based on time, it is usually not safe to assume that a state of control exists unless the plotted points for at least 25 consecutive subgroups fall within 3-sigma control limits. On the other hand, the number of subgroups needed to detect a lack of statistical control at the start may be as small as 4 or 5. Such a precaution against overlooking trouble may increase the risk of a false indication of lack of control. But it is a risk almost always worth taking in order to detect trouble early.

What does this mean? If the objective of a control chart is to detect a process change, and that we want to know how to improve the process, then it would be desirable to assume a larger alpha [α] risk (smaller beta [β] risk) by using control limits smaller than 3 standard deviations from the centerline. This would imply that there would be more false signals of a process change if the process were actually in control. Conversely, if the alpha risk is too small, by using control limits larger than 2 standard deviations from the centerline, then we may not be able to detect a process change when it occurs, which results in a larger beta risk.

Typically, in a process improvement effort, it is desirable to consider a larger alpha risk with a smaller beta risk. However, if the primary objective is to control the process with a minimum of false alarms, then it would be desirable to have a smaller alpha risk with a larger beta risk. The latter situation is preferable if the user is concerned about the occurrence of too many false alarms, and is confident that the control chart limits are the best approximation of chance cause variation.

Once statistical control of the process has been established, occurrence of one plotted point beyond 3-sigma limits in 35 consecutive subgroups or two points in 100 subgroups need not be considered a cause for action.

Note

In a number of examples in **PART 3**, fewer than 25 points are plotted. In most of these examples, evidence of a lack of control is found. In others, it is considered only that the charts fail to show such evidence, and it is not safe to assume a state of statistical control exists.

CONTROL—NO STANDARD GIVEN

3.7. INTRODUCTION

Sections 3.7 to 3.17 cover the technique of analysis for control when no standard is given, as noted under *A* in Section 3.3. Here standard values of μ , σ , p' , etc., are *not given*, hence values derived from the numerical observations are used in arriving at central lines and control limits. This is the situation that exists when the problem at hand is the analysis and

presentation of a given set of experimental data. This situation is also met in the initial stages of a program using the control chart method for controlling quality during production. Available information regarding the quality level and variability resides in the data *to be analyzed* and the central lines and control limits are based on values derived from those data. For a contrasting situation, see Section 3.18.

3.8. CONTROL CHARTS FOR AVERAGES, \bar{X} , AND FOR STANDARD DEVIATIONS, s —LARGE SAMPLES

This section assumes that a set of observed values of a variable X can be subdivided into k rational subgroups (samples), each subgroup containing n of *more than 25* observed values.

A. Large Samples of Equal Size

For samples of size n , the control chart lines are as shown in Table 2.

where

$$\bar{\bar{X}} = \text{the grand average of observed values of } X \text{ for } k \text{ samples,} \quad (3)$$

$$= (\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_k)/k$$

$$\bar{s} = \text{the average subgroup standard deviation,} \quad (4)$$

$$= (s_1 + s_2 + \cdots + s_k)/k$$

where the subscripts 1, 2, ..., k refer to the k subgroups, respectively, all of size n . (For a discussion of this formula, see Supplement 3.B, *Note 3*; also see *Example 1*).

B. Large Samples of Unequal Size

Use Eqs 1 and 2 but compute $\bar{\bar{X}}$ and \bar{s} as follows

$$\begin{aligned} \bar{\bar{X}} &= \text{the grand average of the observed values of } X \text{ for all samples} \\ &= \frac{n_1\bar{X}_1 + n_2\bar{X}_2 + \cdots + n_k\bar{X}_k}{n_1 + n_2 + \cdots + n_k} \end{aligned} \quad (5)$$

= grand total of X values divided by their total number

$$\begin{aligned} \bar{s} &= \text{the weighted standard deviation} \\ &= \frac{n_1s_1 + n_2s_2 + \cdots + n_k s_k}{n_1 + n_2 + \cdots + n_k} \end{aligned} \quad (6)$$

TABLE 2—Equations for Control Chart Lines¹

	Central Line	Control Limits
For averages \bar{X}	$\bar{\bar{X}}$	$\bar{\bar{X}} \pm 3 \frac{\bar{s}}{\sqrt{n-0.5}} \quad (1)^a$
For standard deviations s	\bar{s}	$\bar{s} \pm 3 \frac{\bar{s}}{\sqrt{2n-2.5}} \quad (2)^b$

¹ Previous editions of this manual had used n instead of $n - 0.5$ in Eq 1, and $2(n - 1)$ instead of $2n - 2.5$ in Eq 2 for control limits. Both formulas are approximations, but the present ones are better for n less than 50. Also, it is important to note that the lower control limit for the standard deviation chart is the maximum of $\bar{s} - 3\bar{s}$, and 0 since negative values have no meaning. This idea also applies to the lower control limits for attribute control charts.

^a Eq 1 for control limits is an approximation based on Eq 70, Supplement 3.A. It may be used for n of 10 or more.

^b Eq 2 for control limits is an approximation based on Eq 75, Supplement 3.A. It may be used for n of 10 or more.

TABLE 3—Equations for Control Chart Lines^a

	Central Line	Control Limits	
		Equation Using Factors in Table 6	Alternate Equation
For averages \bar{X}	$\bar{\bar{X}}$	$\bar{\bar{X}} \pm A_3 \bar{s}$	$\bar{\bar{X}} \pm 3 \frac{\bar{s}}{\sqrt{n-0.5}} \quad (7)^a$
For standard deviations s	\bar{s}	$B_4 \bar{s} \text{ and } B_3 \bar{s}$	$\bar{s} \pm 3 \frac{\bar{s}}{\sqrt{2n-2.5}} \quad (8)^b$

^a Alternate Eq 7 is an approximation based on Eq 70, Supplement 3.A. It may be used for n of 10 or more. The values of A_3 in the tables were computed from Eqs 42 and 57 in Supplement 3.A.

^b Alternate Eq 8 is an approximation based on Eq 75, Supplement 3.A. It may be used for n of 10 or more. The values of B_3 and B_4 in the tables were computed from Eqs 42, 61, and 62 in Supplement 3.A.

where the subscripts 1, 2, ..., k refer to the k subgroups, respectively. (For a discussion of this formula, see Supplement 3.B, *Note 3*.) Then compute control limits for each sample size separately, using the individual sample size, n , in the formula for control limits (see *Example 2*).

When most of the samples are of approximately equal size, computing and plotting effort can be saved by the procedure given in Supplement 3.B, *Note 4*.

3.9. CONTROL CHARTS FOR AVERAGES \bar{X} , AND FOR STANDARD DEVIATIONS, s —SMALL SAMPLES

This section assumes that a set of observed values of a variable X is subdivided into k rational subgroups (samples), each subgroup containing $n = 25$ or fewer observed values.

A. Small Samples of Equal Size

For samples of size n , the control chart lines are shown in Table 3. The centerlines for these control charts are defined as the overall average of the statistics being plotted, and can be expressed as

$$\begin{aligned} \bar{\bar{X}} &= \text{the grand average of observed values of} \\ &\quad X \text{ for all samples,} \\ \bar{s} &= \frac{s_1 + s_2 + \cdots + s_k}{k} \end{aligned} \quad (9)$$

and s_1, s_2 , etc., refer to the observed standard deviations for the first, second, etc., samples and factors c_4, A_3, B_3 , and B_4 are given in Table 6. For a discussion of Eq 9, see Supplement 3.B, *Note 3*; also see *Example 3*.

B. Small Samples of Unequal Size

For *small samples of unequal size*, use Eqs 7 and 8 (or corresponding factors) for computing control chart lines. Compute $\bar{\bar{X}}$ by Eq 5. Obtain separate derived values of \bar{s} for the different sample sizes by the following working rule: Compute $\hat{\sigma}$ the overall *average* value of the observed ratio s/c_4 for the individual samples; then compute $\bar{s} = c_4 \hat{\sigma}$ for each sample size n . As shown in *Example 4*, the computation can be simplified by combining in separate groups all samples having the same sample size n . Control limits may then be determined separately for each sample size. These difficulties can be avoided by planning the collection of data so that the samples are made of equal size. The factor c_4 is given in Table 6 (see *Example 4*).

3.10. CONTROL CHARTS FOR AVERAGES, \bar{X} , AND FOR RANGES, R —SMALL SAMPLES

This section assumes that a set of observed values of a variable X is subdivided into k rational subgroups (samples), each subgroup containing $n = 10$ or fewer observed values.

TABLE 4—Equations for Control Chart Lines

	Central Line	Control Limits	
		Equation Using Factors in Table 6	Alternate Equation
For averages \bar{X}	$\bar{\bar{X}}$	$\bar{\bar{X}} \pm A_2 \bar{R}$	$\bar{\bar{X}} \pm 3 \frac{\bar{R}}{d_2 \sqrt{n}} \quad (10)$
For ranges R	\bar{R}	$D_4 \bar{R} \text{ and } D_3 \bar{R}$	$\bar{R} \pm 3 \frac{d_3 \bar{R}}{d_2} \quad (11)$

TABLE 5—Equations for Control Chart Lines^a

	Central Line	Control Limits
Averages using s	$\bar{\bar{X}}$	$\bar{\bar{X}} \pm A_3 \bar{s} \quad (\bar{s} \text{ as given by Eq 9})$
Averages using R	$\bar{\bar{X}}$	$\bar{\bar{X}} \pm A_2 \bar{R} \quad (\bar{R} \text{ as given by Eq 12})$
Standard deviations	\bar{s}	$B_4 \bar{s} \text{ and } B_3 \bar{s} \quad (\bar{s} \text{ as given by Eq 9})$
Ranges	\bar{R}	$D_4 \bar{R} \text{ and } D_3 \bar{R} \quad (\bar{R} \text{ as given by Eq 12})$

^a Control—no standard given (μ, σ , not given)—small samples of equal size.

TABLE 6—Factors for Computing Control Chart Lines—No Standard Given

	Chart for Averages		Chart for Standard Deviations			Chart for Ranges		
	Factors for Control Limits		Factors for Central Line	Factors for Control Limits		Factors for Central Line	Factors for Control Limits	
Observations in Sample, n	A_2	A_3	c_4	B_3	B_4	d_2	D_3	D_4
2	1.880	2.659	0.7979	0	3.267	1.128	0	3.267
3	1.023	1.954	0.8862	0	2.568	1.693	0	2.575
4	0.729	1.628	0.9213	0	2.266	2.059	0	2.282
5	0.577	1.427	0.9400	0	2.089	2.326	0	2.114
6	0.483	1.287	0.9515	0.030	1.970	2.534	0	2.004
7	0.419	1.182	0.9594	0.118	1.882	2.704	0.076	1.924
8	0.373	1.099	0.9650	0.185	1.815	2.847	0.136	1.864
9	0.337	1.032	0.9693	0.239	1.761	2.970	0.184	1.816
10	0.308	0.975	0.9727	0.284	1.716	3.078	0.223	1.777
11	0.285	0.927	0.9754	0.321	1.679	3.173	0.256	1.744
12	0.266	0.886	0.9776	0.354	1.646	3.258	0.283	1.717
13	0.249	0.850	0.9794	0.382	1.618	3.336	0.307	1.693
14	0.235	0.817	0.9810	0.406	1.594	3.407	0.328	1.672
15	0.223	0.789	0.9823	0.428	1.572	3.472	0.347	1.653
16	0.212	0.763	0.9835	0.448	1.552	3.532	0.363	1.637
17	0.203	0.739	0.9845	0.466	1.534	3.588	0.378	1.622
18	0.194	0.718	0.9854	0.482	1.518	3.640	0.391	1.609
19	0.187	0.698	0.9862	0.497	1.503	3.689	0.404	1.596
20	0.180	0.680	0.9869	0.510	1.490	3.735	0.415	1.585
21	0.173	0.663	0.9876	0.523	1.477	3.778	0.425	1.575
22	0.167	0.647	0.9882	0.534	1.466	3.819	0.435	1.565
23	0.162	0.633	0.9887	0.545	1.455	3.858	0.443	1.557
24	0.157	0.619	0.9892	0.555	1.445	3.895	0.452	1.548
25	0.153	0.606	0.9896	0.565	1.435	3.931	0.459	1.541
Over 25	...	a	b	c	d
$^a 3/\sqrt{n-0.5}$		$^c 1 - 3/\sqrt{2n-2.5}$						
$^b (4n-4)/(4n-3)$		$^d 1 + 3/\sqrt{2n-2.5}$						

The range, R , of a sample is the difference between the largest observation and the smallest observation. When $n = 10$ or less, simplicity and economy of effort can be obtained by using control charts for \bar{X} and R in place of control charts for \bar{X} and s . The range is not recommended, however, for samples of more than 10 observations, since it becomes rapidly less effective than the standard deviation as a detector of assignable causes as n increases beyond this value. In some circumstances, it may be found satisfactory to use the control chart for ranges for samples up to $n = 15$, as when data are plentiful or cheap. On occasion, it may be desirable

to use the chart for ranges for even larger samples: for this reason Table 6 gives factors for samples as large as $n = 25$.

A. Small Samples of Equal Size

For samples of size n , the control chart lines are as shown in Table 4.

Where $\bar{\bar{X}}$ is the grand average of observed values of X for *all* samples, \bar{R} is the average value of range R for the k individual samples

$$(\bar{R}_1 + \bar{R}_2 + \cdots + \bar{R}_k)/k \quad (12)$$

and the factors d_2 , A_2 , D_3 , and D_4 are given in Table 6 and d_3 in Table 49 (see *Example 5*).

B. Small Samples of Unequal Size

For *small samples of unequal size*, use Eqs 10 and 11 (or corresponding factors) for computing control chart lines. Compute $\bar{\bar{X}}$ by Eq 5. Obtain separate derived values of \bar{R} for the different sample sizes by the following working rule: compute $\hat{\sigma}$, the overall *average* value of the observed ratio R/d_2 for the individual samples. Then compute $\bar{R} = d_2\hat{\sigma}$ for each sample size n . As shown in *Example 6*, the computation can be simplified by combining in separate groups all samples having the same sample size n . Control limits may then be determined separately for each sample size. These difficulties can be avoided by planning the collection of data so that the samples are made of equal size.

3.11. SUMMARY, CONTROL CHARTS FOR \bar{X} , s , AND r —NO STANDARD GIVEN

The most useful formulas and equations from Sections 3.7 to 3.10, inclusive, are collected in Table 5 and are followed by Table 6, which gives the factors used in these and other formulas.

3.12. CONTROL CHARTS FOR ATTRIBUTES DATA

Although in what follows the fraction p is designated *fraction nonconforming*, the methods described can be applied quite generally and p may in fact be used to represent the ratio of the number of items, occurrences, etc. that possess some given attribute to the total number of items under consideration.

The fraction nonconforming, p , is particularly useful in analyzing inspection and test results that are obtained on a “go/no-go” basis (method of attributes). In addition, it is used in analyzing results of measurements that are made on a scale and recorded (method of variables). In the latter case, p may be used to represent the fraction of the total number of measured values falling above any limit, below any limit, between any two limits, or outside any two limits.

The fraction p is used widely to represent the fraction nonconforming, that is, the ratio of the number of nonconforming units (articles, parts, specimens, etc.) to the total number of units under consideration. The fraction nonconforming is used as a measure of quality with respect to a single quality characteristic or with respect to two or more quality characteristics treated collectively. In this connection it is important to distinguish between a *nonconformity* and a *nonconforming unit*. A nonconformity is a *single* instance of a failure to meet some requirement, such as a failure to comply with a particular requirement imposed on a unit of product with respect to a single quality characteristic. For example, a unit containing departures from requirements of the drawings and specifications with respect to (1) a particular dimension, (2) finish, and (3) absence of chamfer, contains three defects. The words “nonconforming unit” define a unit (article, part, specimen, etc.) containing one or more “nonconformities” with respect to the quality characteristic under consideration.

When only a single quality characteristic is under consideration, or when only one nonconformity can occur on a unit, the number of nonconforming units in a sample will equal the number of nonconformities in that sample.

However, it is suggested that under these circumstances the phrase “number of nonconforming units” be used rather than “number of nonconformities.”

Control charts for attributes are usually based either on counts of occurrences or on the average of such counts. This means that a series of attribute samples may be summarized in one of these two principal forms of a control chart and, although they differ in appearance, both will produce essentially the same evidence as to the state of statistical control. Usually it is not possible to construct a second type of control chart based on the same attribute data, which gives evidence different from that of the first type of chart as to the state of statistical control in the way the \bar{X} and s (or \bar{X} and R) control charts do for variables.

An exception may arise when, say, samples are composed of similar units in which various numbers of nonconformities may be found. If these numbers in individual units are recorded, then in principle it is possible to plot a second type of control chart reflecting variations in the number of nonuniformities from unit to unit within samples. Discussion of statistical methods for helping to judge whether this second type of chart gives different information on the state of statistical control is beyond the scope of this Manual.

In control charts for attributes, as in s and R control charts for small samples, the lower control limit is often at or near zero. A point above the upper control limit on an attribute chart may lead to a costly search for cause. It is important, therefore, especially when small counts are likely to occur, that the calculation of the upper limit accounts adequately for the magnitude of chance variation that may be expected. Ordinarily there is little to justify the use of a control chart for attributes if the occurrence of one or two nonconformities in a sample causes a point to fall above the upper control limit.

Note

To avoid or minimize this problem of small counts, it is best if the expected or estimated number of occurrences in a sample is four or more. An attribute control chart is least useful when the expected number of occurrences in a sample is less than one.

Note

The lower control limit based on the formulas given may result in a negative value that has no meaning. In such situations, the lower control limit is simply set at zero.

It is important to note that a positive non-zero lower control limit offers the opportunity for a plotted point to fall below this limit when the process quality level significantly improves. Identifying the assignable cause(s) for such points will usually lead to opportunities for process and quality improvements.

3.13. CONTROL CHART FOR FRACTION NONCONFORMING, p

This section assumes that the total number of units tested is subdivided into k rational subgroups (samples) consisting of n_1 , n_2 , ..., n_k units, respectively, for each of which a value of p is computed.

Ordinarily the control chart of p is most useful when the samples are large, say when n is 50 or more, and when the expected number of nonconforming units (or other

TABLE 7—Equations for Control Chart Lines

	Central Line	Control Limits
For values of p	\bar{p}	$\bar{p} \pm 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$ (14)

occurrences of interest) per sample is four or more, that is, the expected np is four or more. When n is less than 25 or when the expected np is less than 1, the control chart for p may not yield reliable information on the state of control.

The average fraction nonconforming \bar{p} is defined as

$$\begin{aligned}\bar{p} &= \frac{\text{total \# of nonconforming units in all samples}}{\text{total \# of units in all samples}} \\ &= \text{fraction nonconforming in the complete set of test results.}\end{aligned}\quad (13)$$

A. Samples of Equal Size

For a sample of size n , the control chart lines are as follows in Table 7 (see *Example 7*).

When \bar{p} is small, say less than 0.10, the factor $1 - \bar{p}$ may be replaced by unity for most practical purposes, which gives control limits for p by the simple relation

$$\bar{p} \pm 3\sqrt{\frac{\bar{p}}{n}} \quad (14a)$$

B. Samples of Unequal Size

Proceed as for samples of equal size but compute control limits for each sample size separately.

When the data are in the form of a series of k subgroup values of p and the corresponding sample sizes n , \bar{p} may be computed conveniently by the relation

$$\bar{p} = \frac{n_1 p_1 + n_2 p_2 + \cdots + n_k p_k}{n_1 + n_2 + \cdots + n_k} \quad (15)$$

where the subscripts 1, 2, ..., k refer to the k subgroups. When most of the samples are of approximately equal size, computation and plotting effort can be saved by the procedure in Supplement 3.B, *Note 4* (see *Example 8*).

Note

If a sample point falls above the upper control limit for p when $n\bar{p}$ is less than 4, the following check and adjustment method is recommended to reduce the incidence of misleading indications of a lack of control. If the non-integral remainder of the product of n and the upper control limit value for p is one-half or less, the indication of a lack of control stands. If that remainder exceeds one-half, add one to the product and divide the sum by n to calculate an adjusted upper control limit for p . Check for an indication of lack of control in p against this adjusted limit (see *Examples 7* and *8*).

3.14. CONTROL CHART FOR NUMBERS OF NONCONFORMING UNITS, np

The control chart for np , number of conforming units in a sample of size n , is the equivalent of the control chart for p ,

TABLE 8—Equations for Control Chart Lines

	Central Line	Control Limits
For values of np	$n\bar{p}$	$n\bar{p} \pm 3\sqrt{n\bar{p}(1-\bar{p})}$ (16)

for which it is a convenient practical substitute when all samples have the same size n . It makes direct use of the number of nonconforming units np , in a sample (np = the fraction nonconforming times the sample size).

For samples of size n , the control chart lines are as shown in Table 8, where

$$\begin{aligned}n\bar{p} &= \text{total number of nonconforming units in all samples/number of samples} \\ &= \text{the average number of nonconforming units in the } k \text{ individual samples, and} \\ \bar{p} &= \text{the value given by Eq 13.}\end{aligned}\quad (17)$$

When \bar{p} is *small*, say, less than 0.10, the factor $1 - \bar{p}$ may be replaced by unity for most practical purposes, which gives control limits for np by the simple relation

$$n\bar{p} \pm 3\sqrt{n\bar{p}} \quad (18)$$

or, in other words, it can be read as the avg. number of nonconforming units $\pm 3\sqrt{\text{average number of nonconforming units}}$ where "average number of nonconforming units" means the average number in samples of equal size (see *Example 7*).

When the sample size, n , varies from sample to sample, the control chart for p (Section 3.13) is recommended in preference to the control chart for np ; in this case, a graphical presentation of values of np does not give an easily understood picture, since the expected values, $n\bar{p}$, (central line on the chart) vary with n , and therefore the plotted values of np become more difficult to compare. The recommendations of Section 3.13 as to size of n and expected np in a sample apply also to control charts for the numbers of nonconforming units.

When only a single quality characteristic is under consideration, and when only one nonconformity can occur on a unit, the word "nonconformity" can be substituted for the words "nonconforming unit" throughout the discussion of this section but this practice is not recommended.

Note

If a sample point falls above the upper control limit for np when $n\bar{p}$ is less than 4, the following check and adjustment procedure is to be recommended to reduce the incidence of misleading indications of a lack of control. If the nonintegral remainder of the upper control limit value for np is one-half or less, the indication of a lack of control stands. If that remainder exceeds one-half, add one to the upper control limit value for np to adjust it. Check for an indication of lack of control in np against this adjusted limit (see *Example 7*).

3.15. CONTROL CHART FOR NONCONFORMITIES PER UNIT, u

In inspection and testing, there are circumstances where it is possible for several nonconformities to occur on a single unit (article, part, specimen, unit length, unit area, etc.) of product, and it is desired to control the number of

nonconformities per unit, rather than the fraction nonconforming. For any given sample of units, the numerical value of nonconformities per unit, u , is equal to the number of nonconformities in all the units in the sample divided by the number of units in the sample.

The control chart for the nonconformities per unit in a sample u is convenient for a product composed of units for which inspection covers more than one characteristic, such as dimensions checked by gages, electrical and mechanical characteristics checked by tests, and visual nonconformities observed by the eye. Under these circumstances, several independent nonconformities may occur on one unit of product and a better measure of quality is obtained by making a count of all nonconformities observed and dividing by the number of units inspected to give a value of nonconformities per unit, rather than merely counting the number of nonconforming units to give a value of fraction nonconforming. This is particularly the case for complex assemblies where the occurrence of two or more nonconformities on a unit may be relatively frequent. However, only *independent* nonconformities are counted. Thus, if two nonconformities occur on one unit of product and the second is caused by the first, only the first is counted.

The control chart for nonconformities per unit (more particularly the chart for number of nonconformities, see Section 3.16) is a particularly convenient one to use when the number of possible nonconformities on a unit is indeterminate, as for physical defects (finish or surface irregularities, flaws, pin-holes, etc.) on such products as textiles, wire, sheet materials, etc., which are not continuous or extensive. Here, the opportunity for nonconformities may be numerous though the chances of nonconformities occurring at any one spot may be small.

This section assumes that the total number of units tested is subdivided into k rational subgroups (samples) consisting of n_1, n_2, \dots, n_k units, respectively, for each of which a value of u is computed.

The control chart for u is most useful when the expected nu is 4 or more. When the expected nu is less than 1, the control chart for u may not yield reliable information on the state of control.

The average nonconformities per unit, \bar{u} is defined as

$$\begin{aligned}\bar{u} &= \frac{\text{total \# nonconformities in all samples}}{\text{total \# units in all samples}} \\ &= \text{nonconformities per unit in the complete set of test results}\end{aligned}\quad (19)$$

The simplified relations shown for control limits for nonconformities per unit assume that for each of the characteristics under consideration the ratio of the expected number of nonconformities to the possible number of nonconformities is small, say less than 0.10, an assumption that is commonly satisfied in quality control work. For an explanation of the nature of the distribution involved, see Supplement 3.B, *Note 5*.

A. Samples of Equal Size

For samples of size n (n = number of units), the control chart lines are as shown in Table 9.

For samples of equal size, a chart for the number of nonconformities, c , is recommended, see Section 3.16. In the special case where each sample consists of only one unit, that is, $n = 1$,

TABLE 9—Equations for Control Chart Lines

	Central Line	Control Limits
For values of u	\bar{u}	$\bar{u} \pm 3\sqrt{\frac{\bar{u}}{n}} \quad (20)$

then the chart for u (nonconformities per unit) is identical with that chart for c (number of nonconformities) and may be handled in accordance with Section 3.16. In this case the chart may be referred to either as a chart for nonconformities per unit or as a chart for number of nonconformities, but the latter designation is recommended (see *Example 9*).

B. Samples of Unequal Size

Proceed as for samples of equal size but compute the control limits for each sample size separately.

When the data are in the form of a series of subgroup values of u and the corresponding sample sizes, \bar{u} may be computed by the relation

$$\bar{u} = \frac{n_1 u_1 + n_2 u_2 + \dots + n_k u_k}{n_1 + n_2 + \dots + n_k} \quad (21)$$

where as before, the subscripts 1, 2, ..., k refer to the k subgroups.

Note that n_1, n_2 , etc., need not be whole numbers. For example, if u represents nonconformities per 1,000 ft of wire, samples of 4,000 ft, 5,280 ft, etc., then the corresponding values will be 4.0, 5.28, etc., units of 1,000 ft.

When most of the samples are of approximately equal size, computing and plotting effort can be saved by the procedure in Supplement 3.B, *Note 4* (see *Example 10*).

Note

If a sample point falls above the upper limit for u where $n\bar{u}$ is less than 4, the following check and adjustment procedure is recommended to reduce the incidence of misleading indications of a lack of control. If the nonintegral remainder of the product of n and the upper control limit value for u is one half or less, the indication of a lack of control stands. If that remainder exceeds one-half, add one to the product and divide the sum by n to calculate an adjusted upper control limit for u . Check for an indication of lack of control in u against this adjusted limit (see *Examples 9* and *10*).

3.16. CONTROL CHART FOR NUMBER OF NONCONFORMITIES, c

The control chart for c , the number of nonconformities in a sample, is the equivalent of the control chart for u , for which it is a convenient practical substitute when all samples have the *same size* n (number of units).

A. Samples of Equal Size

For samples of equal size, if the average number of nonconformities per sample is \bar{c} the control chart lines are as shown in Table 10.

TABLE 10—Equations for Control Chart Lines

	Central Line	Control Limits
For values of c	\bar{c}	$\bar{c} \pm 3\sqrt{\bar{c}} \quad (22)$

where

$$\bar{c} = \frac{\text{total number of nonconformities in all samples}}{\text{number of samples}} \quad (23)$$

= average number of nonconformities per sample.

The use of c is especially convenient when there is no natural unit of product, as for nonconformities over a surface or along a length, and where the problem is to determine uniformity of quality in equal lengths, areas, etc., of product (see *Examples 9 and 11*).

B. Samples of Unequal Size

For samples of unequal size, first compute the average nonconformities per unit \bar{u} by Eq 19; then compute the control limits for each sample size separately as shown in Table 11.

The control chart for u is recommended as preferable to the control chart for c when the sample size varies from sample to sample for reasons stated in discussing the control charts for p and np . The recommendations of Section 3.15 as to expected $c = n\bar{u}$ also applies to control charts for numbers of nonconformities.

Note

If a sample point falls above the upper control limit for c when $n\bar{u}$ is less than 4, the following check and adjustment procedure is to be recommended to reduce the incidence of misleading indications of a lack of control. If the nonintegral remainder of the upper control limit for c is one-half or less, the indication of a lack of control stands. If that remainder exceeds one-half, add one to the upper control limit value for c to adjust it. Check for an indication of lack of control in c against this adjusted limit (see *Examples 9 and 11*).

3.17. SUMMARY, CONTROL CHARTS FOR p , np , u , AND c —NO STANDARD GIVEN

The formulas of Sections 3.13 to 3.16, inclusive, are collected as shown in Table 12 for convenient reference.

TABLE 11—Equations for Control Chart Lines		
	Central Line	Control Limits
For values of c	$n\bar{u}$	$n\bar{u} \pm 3\sqrt{n\bar{u}} \quad (24)$

CONTROL WITH RESPECT TO A GIVEN STANDARD

3.18. INTRODUCTION

Sections 3.18 to 3.27 cover the technique of analysis for control with respect to a given standard, as noted under (B) in Section 3.3. Here, standard values of μ , σ , p , etc., are *given*, and are those corresponding to a given standard distribution. These standard values, designated as μ_0 , σ_0 , p_0 , etc., are used in calculating both central lines and control limits. (When only μ_0 is given and no prior data are available for establishing a value of σ_0 , analyze data from the first production period as in Sections 3.7 to 3.10, but use μ_0 for the central line.)

Such standard values are usually based on a control chart analysis of previous data (for the details, see Supplement 3.B, *Note 6*), but may be given on the basis described in Section 3.3B. Note that these standard values are set up *before* the detailed analysis of the data at hand is undertaken and frequently before the data to be analyzed are collected. In addition to the standard values, only the information regarding sample size or sizes is required in order to compute central lines and control limits.

For example, the values to be used as central lines on the control charts are

for averages, μ_0
 for standard deviations, $c_4\sigma_0$
 for ranges, $d_2\sigma_0$
 for values of p , p_0
 etc.

where factors c_4 and d_2 , which depend only on the sample size, n , are given in Table 16, and defined in Supplement 3.A.

Note that control with respect to a given standard may be a more exacting requirement than control with no standard given, described in Sections 3.7 to 3.17. The data must exhibit not only control but control at a standard level and with no more than standard variability.

Extending control limits obtained from a set of existing data into the future and using these limits as a basis for purposive control of quality during production, is equivalent to adopting, as standard, the values obtained from the existing data. Standard values so obtained may be tentative and subject to revision as more experience is accumulated (for details, see Supplement 3.B, *Note 6*).

TABLE 12—Equations for Control Chart Lines			
Control—No Standard Given—Attributes Data			
	Central Line	Control Limits	Approximation
Fraction nonconforming p	\bar{p}	$\bar{p} \pm 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$	$\bar{p} \pm 3\sqrt{\frac{\bar{p}}{n}}$
Number of nonconforming units np	$n\bar{p}$	$n\bar{p} \pm 3\sqrt{n\bar{p}(1-\bar{p})}$	$n\bar{p} \pm 3\sqrt{n\bar{p}}$
Nonconformities per unit, u	\bar{u}	$\bar{u} \pm 3\sqrt{\frac{\bar{u}}{n}}$...
Number of nonconformities c			
samples of equal size	\bar{c}	$\bar{c} \pm 3\sqrt{\bar{c}}$...
samples of unequal size	$n\bar{u}$	$n\bar{u} \pm 3\sqrt{n\bar{u}}$...

TABLE 13—Equations for Control Chart Lines²

		Control Limits	
		Formula Using Factors in Table 16	Alternate Formula
For averages \bar{X}	μ_0	$\mu_0 \pm A\sigma_0$	$\mu_0 \pm 3\frac{\sigma_0}{\sqrt{n}}$ (25)
For standard deviations s	$c_4\sigma_0$	$B_6\sigma_0$ and $B_4\sigma_0$	$c_4\sigma_0 \pm \frac{\sigma_0}{\sqrt{2n-1.5}}$ (26) ^a

² Previous editions of this manual had $2(n-1)$ instead of $2n-1.5$ in alternate Eq 26. Both formulas are approximations, but the present one is better for n less than 50.

^a Alternate Eq 26 is an approximation based on Eq 74, Supplement 3.A. It may be used for n of 10 or more. The values of B_5 and B_6 given in the tables are computed from Eqs 42, 59, and 60 in Supplement 3.A.

Note

Two situations that are not covered specifically within this section should be mentioned.

1. In some cases a standard value of μ is given as noted above, but no standard value is given for σ . Here σ is estimated from the analysis of the data at hand and the problem is essentially one of controlling \bar{X} at the standard level μ_0 that has been given.
2. In other cases, interest centers on controlling the conformance to specified minimum and maximum limits within which material is considered acceptable, sometimes established without regard to the actual variation experienced in production. Such limits may prove unrealistic when data are accumulated and an estimate of the standard deviation, say σ^* of the process is obtained therefrom. If the natural spread of the process (a band having a width of $6\sigma^*$), is wider than the spread between the specified limits, it is necessary either to adjust the specified limits or to operate within a band narrower than the process capability. Conversely, if the spread of the process is narrower than the spread between the specified limits, the process will deliver a more uniform product than required. Note that in the latter event when only maximum and minimum limits are specified, the process can be operated at a level above or below the indicated mid-value without risking the production of significant amounts of unacceptable material.

3.19. CONTROL CHARTS FOR AVERAGES \bar{X} AND FOR STANDARD DEVIATION, s

For samples of size n , the control chart lines are as shown in Table 13.

For samples of n greater than 25, replace c_4 by $(4n-4)/(4n-3)$.

See Examples 12 and 13; also see Supplement 3.B, Note 9.

For samples of $n = 25$ or less, use Table 16 for factors A , B_5 , and B_6 . Factors c_4 , A , B_5 , and B_6 are defined in Supplement 3.A. See Examples 14 and 15.

3.20. CONTROL CHART FOR RANGES R

The range, R , of a sample is the difference between the largest observation and the smallest observation.

For samples of size n , the control chart lines are as shown in Table 14.

Use Table 16 for factors d_2 , D_1 , and D_2 .

Factors d_2 , d_3 , D_1 , and D_2 are defined in Supplement 3.A.

For comments on the use of the control chart for ranges, see Section 3.10 (also see Example 16).

3.21. SUMMARY, CONTROL CHARTS FOR \bar{X} , s , AND r —STANDARD GIVEN

The most useful formulas from Sections 3.19 and 3.20 are summarized as shown in Table 15 and are followed by Table 16, which gives the factors used in these and other formulas.

3.22. CONTROL CHARTS FOR ATTRIBUTES DATA

The definitions of terms and the discussions in Sections 3.12 to 3.16, inclusive, on the use of the fraction nonconforming, p , number of nonconforming units, np , nonconformities per unit, u , and number of nonconformities, c , as measures of quality are equally applicable to the sections which follow and will not be repeated here. It will suffice to discuss the central lines and control limits when standards are given.

3.23. CONTROL CHART FOR FRACTION NONCONFORMING, p

Ordinarily, the control chart for p is most useful when samples are large, say, when n is 50 or more and when the expected number of nonconforming units (or other occurrences of interest) per sample is four or more, that is, the expected values of np is four or more. When n is less than

TABLE 15—Equations for Control Chart Lines

Control with Respect to a Given Standard (μ_0 , σ_0 Given)		
	Central Line	Control Limits
Average \bar{X}	μ_0	$\mu_0 \pm A\sigma_0$
Standard deviation s	$c_4\sigma_0$	$B_6\sigma_0$ and $B_5\sigma_0$
Range R	$d_2\sigma_0$	$D_2\sigma_0$ and $D_1\sigma_0$

TABLE 14—Equations for Control Chart Lines

	Central Line	Control Limits	
		Equation Using Factors in Table 16	Alternate Equation
For range R	$d_2\sigma_0$	$D_2\sigma_0$ and $D_1\sigma_0$	$d_2\sigma_0 \pm d_3\sigma_0$ (27)

TABLE 16—Factors for Computing Control Chart Lines—Standard Given

	Chart for Averages	Chart for Standard Deviations			Chart for Ranges		
	Factors for Control Limits	Factor for Central Line	Factors for Control Limits		Factor for Central Line	Factors for Control Limits	
Observations in Sample, n	A	C_4	B_5	B_6	d_2	D_1	D_2
2	2.121	0.7979	0	2.606	1.128	0	3.686
3	1.732	0.8862	0	2.276	1.693	0	4.358
4	1.500	0.9213	0	2.088	2.059	0	4.698
5	1.342	0.9400	0	1.964	2.326	0	4.918
6	1.225	0.9515	0.029	1.874	2.534	0	5.079
7	1.134	0.9594	0.113	1.806	2.704	0.205	5.204
8	1.061	0.9650	0.179	1.751	2.847	0.388	5.307
9	1.000	0.9693	0.232	1.707	2.970	0.547	5.393
10	0.949	0.9727	0.276	1.669	3.078	0.686	5.469
11	0.905	0.9754	0.313	1.637	3.173	0.811	5.535
12	0.866	0.9776	0.346	1.610	3.258	0.923	5.594
13	0.832	0.9794	0.374	1.585	3.336	1.025	5.647
14	0.802	0.9810	0.399	1.563	3.407	1.118	5.696
15	0.775	0.9823	0.421	1.544	3.472	1.203	5.740
16	0.750	0.9835	0.440	1.526	3.532	1.282	5.782
17	0.728	0.9845	0.458	1.511	3.588	1.356	5.820
18	0.707	0.9854	0.475	1.496	3.640	1.424	5.856
19	0.688	0.9862	0.490	1.483	3.689	1.489	5.889
20	0.671	0.9869	0.504	1.470	3.735	1.549	5.921
21	0.655	0.9876	0.516	1.459	3.778	1.606	5.951
22	0.640	0.9882	0.528	1.448	3.819	1.660	5.979
23	0.626	0.9887	0.539	1.438	3.858	1.711	6.006
24	0.612	0.9892	0.549	1.429	3.895	1.759	6.032
25	0.600	0.9896	0.559	1.420	3.931	1.805	6.056
Over 25	$3/\sqrt{n}$	a	b	c

^a $(4n - 4)/(4n - 3)$.
^b $(4n - 4)/(4n - 3) - 3/\sqrt{2n - 2.5}$
^c $(4n - 4)/(4n - 3) + 3/\sqrt{2n - 2.5}$
 See Supplement 3.B, Note 9, on replacing first term in footnotes b and c by unity.

25 or the expected np is less than 1, the control chart for p may not yield reliable information on the state of control even with respect to a given standard.

For samples of size n , where p_0 is the standard value of p , the control chart lines are as shown in Table 17 (see Example 17).

When p_0 is small, say less than 0.10, the factor $1 - p_0$ may be replaced by unity for most practical purposes, which gives the simple relation for computing the control limits for p as

$$p = p_0 \pm 3\sqrt{\frac{p_0}{n}} \quad (28a)$$

TABLE 17—Equations for Control Chart Lines

	Central Line	Control Limits
For values of p	p_0	$p_0 \pm 3\sqrt{\frac{p_0(1-p_0)}{n}}$ (28)

TABLE 18—Equations for Control Chart Lines

	Central Line	Control Limits
For values of np	np_0	$np_0 \pm 3\sqrt{np_0(1-p_0)}$ (29)

For samples of unequal size, proceed as for samples of equal size but compute control limits for each sample size separately (see *Example 18*).

When detailed inspection records are maintained, the control chart for p may be broken down into a number of component charts with advantage (see *Example 19*). See the **NOTE** at the end of Section 3.13 for possible adjustment of the upper control limit when np_0 is less than 4. (Substitute np_0 for $n\bar{p}$.) See *Examples 17, 18, and 19* for applications.

3.24. CONTROL CHART FOR NUMBER OF NONCONFORMING UNITS, np

The control chart for np , number of nonconforming units in a sample, is the equivalent of the control chart for fraction nonconforming, p , for which it is a convenient practical substitute, particularly when all samples have the *same size, n*. It makes direct use of the number of nonconforming units, np , in a sample (np = the product of the sample size and the fraction nonconforming). See *Example 17*.

For samples of size n , where p_0 is the standard value of p , the control chart lines are as shown in Table 18.

When p_0 is *small*, say less than 0.10, the factor $1 - p_0$ may be replaced by unity for most practical purposes, which gives the simple relation for computing the control limits for np as

$$np_0 \pm 3\sqrt{np_0} \quad (30)$$

As noted in Section 3.14, the control chart for p is recommended as preferable to the control chart for np when the sample size varies from sample to sample. The recommendations of Section 3.23 as to size of n and the expected np in a sample also apply to control charts for the number of nonconforming units.

When only a single quality characteristic is under consideration, and when only one nonconformity can occur on a unit, the word “nonconformity” can be substituted for the words “nonconforming unit” throughout the discussion of this article, but this practice is not recommended. See the **NOTE** at the end of Section 3.14 for possible adjustment of the upper control limit when np_0 is less than 4. (Substitute np_0 for $n\bar{p}$.) See *Examples 17 and 18*.

3.25. CONTROL CHART FOR NONCONFORMITIES PER UNIT, u

For samples of size n (n = number of units), where u_0 is the standard value of u , the control chart lines are as shown in Table 19.

See *Examples 20 and 21*.

As noted in Section 3.15, the relations given here assume that for each of the characteristics under consideration, the

ratio of the expected to the possible number of nonconformities is small, say less than 0.10.

If u represents “nonconformities per 1,000 ft of wire,” a “unit” is 1,000 ft of wire. Then if a series of samples of 4,000 ft are involved, u_0 represents the standard or expected number of nonconformities per 1,000 ft, and $n = 4$. Note that n need not be a whole number, for if samples comprise 5,280 ft of wire each, $n = 5.28$, that is, 5.28 units of 1,000 ft (see *Example 11*).

Where each sample consists of only one unit, that is $n = 1$, then the chart for u (nonconformities per unit) is identical with the chart for c (number of nonconformities) and may be handled in accordance with Section 3.26. In this case the chart may be referred to either as a chart for nonconformities per unit or as a chart for number of nonconformities, but the latter practice is recommended.

Ordinarily, the control chart for u is most useful when the expected nu is 4 or more. When the expected nu is less than 1, the control chart for u may not yield reliable information on the state of control even with respect to a given standard.

See the **NOTE** at the end of Section 3.15 for possible adjustment of the upper control limit when nu_0 is less than 4. (Substitute nu_0 for $n\bar{u}$.) See *Examples 20 and 21*.

3.26. CONTROL CHART FOR NUMBER OF NONCONFORMITIES, c

The control chart for c , number of nonconformities in a sample, is the equivalent of the control chart for nonconformities per unit for which it is a convenient practical substitute when all samples have the *same size, n* (number of units). Here c is the *number of nonconformities in a sample*.

If the standard value is expressed in terms of number of nonconformities per sample of some given size, that is, expressed merely as c_0 , and the samples are all of the same given size (same number of product units, same area of opportunity for defects, same sample length of wire, etc.), then the control chart lines are as shown in Table 20.

Use of c_0 is especially convenient when there is no natural unit of product, as for nonconformities over a surface or along a length, and where the problem of interest is to compare uniformity of quality in samples of the same size, no matter how constituted (see *Example 21*).

When the sample size, n (number of units), varies from sample to sample, and the standard value is expressed in terms of nonconformities per unit, the control chart lines are as shown in Table 21.

TABLE 20—Equations for Control Chart Lines (c_0 Given).

	Central Line	Control Limits
For number of nonconformities, c	c_0	$c_0 \pm 3\sqrt{c_0}$ (32)

TABLE 19—Equations for Control Chart Lines

	Central Line	Control Limits
For values of u	u_0	$u_0 \pm 3\sqrt{\frac{u_0}{n}}$ (31)

TABLE 21—Equations for Control Chart Lines (u_0 Given)

	Central Line	Control Limits
For values of c	nu_0	$nu_0 \pm 3\sqrt{nu_0}$ (33)

TABLE 22—Equations for Control Chart Lines

Control with Respect to a Given Standard (p_0 , np_0 , u_0 , or c_0 Given)			
	Central Line	Control Limits	Approximation
Fraction nonconforming, p	p_0	$p_0 \pm 3\sqrt{\frac{p_0(1-p_0)}{n}}$	$p_0 \pm 3\sqrt{\frac{p_0}{n}}$
Number of nonconforming units, np	np_0	$np_0 \pm 3\sqrt{np_0(1-p_0)}$	$np_0 \pm 3\sqrt{np_0}$
Nonconformities per unit, u	u_0	$u_0 \pm 3\sqrt{\frac{u_0}{n}}$	
Number of nonconformities, c Samples of equal size (c_0 given)	c_0	$c_0 \pm 3\sqrt{c_0}$	
Samples of unequal size (u_0 given)	nu_0	$nu_0 \pm 3\sqrt{nu_0}$	

Under these circumstances, the control chart for u (Section 3.25) is recommended in preference to the control chart for c , for reasons stated in Section 3.14 in the discussion of control charts for p and for np . The recommendations of Section 3.25 as to the expected $c = nu$ also applies to control charts for nonconformities.

See the **NOTE** at the end of Section 3.16 for possible adjustment of the upper control limit when nu_0 is less than 4. (Substitute $c_0 = nu_0$ for $n\bar{u}$.) See *Example 21*.

3.27. SUMMARY, CONTROL CHARTS FOR p , np , u , AND c —STANDARD GIVEN

The formulas of Sections 3.22 to 3.26, inclusive, are collected as shown in Table 22 for convenient reference.

CONTROL CHARTS FOR INDIVIDUALS

3.28. INTRODUCTION

Sections 3.28 to 3.30³ deal with control charts for individuals, in which individual observations are plotted one by one. This type of control chart has been found useful more particularly in process control when only one observation is obtained per lot or batch of material or at periodic intervals from a process. This situation often arises when: (a) sampling or testing is destructive, (b) costly chemical analyses or physical tests are involved, and (c) the material sampled at any one time (such as a batch) is normally quite homogeneous, as for a well-mixed fluid or aggregate.

The purpose of such control charts is to discover whether the individual observed values differ from the expected value by an amount greater than should be attributed to chance.

When there is some definite rational basis for grouping the batches or observations into rational subgroups, as, for example, four successive batches in a single shift, the method shown in Section 3.29 may be followed. In this case, the control chart for individuals is merely an adjunct to the more usual charts but will react more quickly to a sharp change in the process than the \bar{X} chart. This may be important when a single batch represents a considerable sum of money.

When there is no definite basis for grouping data, the control limits may be based on the variation between batches, as described in Section 3.30. A measure of this variation is obtained from moving ranges of two observations

each (the absolute value of successive differences between individual observations that are arranged in chronological order).

A control chart for moving ranges may be prepared as a companion to the chart for individuals, if desired, using the formulas of Section 3.30. It should be noted that adjacent moving ranges are correlated, as they have one observation in common.

The methods of Sections 3.29 and 3.30 may be applied appropriately in some cases where more than one observation is obtained per lot or batch, as for example with very homogeneous batches of materials, for instance, chemical solutions, batches of thoroughly mixed bulk materials, etc., for which repeated measurements on a single batch show the within-batch variation (variation of quality within a batch and errors of measurement) to be very small as compared with between-batch variation. In such cases, the *average* of the several observations for a batch may be *treated as an individual observation*. However, this procedure should be used with great caution; the restrictive conditions just cited should be carefully noted.

The control limits given are three sigma control limits in all cases.

3.29. CONTROL CHART FOR INDIVIDUALS, X —USING RATIONAL SUBGROUPS

Here the control chart for individuals is commonly used as an adjunct to the more usual \bar{X} and s , or \bar{X} and R , control charts. This can be useful, for example, when it is important to react immediately to a single point that may be out of statistical control, when the ability to localize the source of an individual point that has gone out of control is important, or when a rational subgroup consisting of more than two points is either impractical or nonsensical. Proceed exactly as in Sections 3.9 to 3.11 (control—no standard given) or Sections 3.19 to 3.21 (control—standard given), whichever is applicable, and prepare control charts for \bar{X} and s , or for \bar{X} and R . In addition, prepare a control chart for individuals having the same central line as the \bar{X} chart but compute the control limits as shown in Table 23.

Table 26 gives values of E_2 and E_3 for samples of $n = 10$ or less. Values that are more complete are given in Table 50, Supplement 3.A for n through 25 (see *Examples 22* and *23*).

³ To be used with caution if the distribution of individual values is markedly asymmetrical.

TABLE 23—Equations for Control Chart Lines

Chart for Individuals—Associated with Chart for s or R Having Sample Size n			
Nature of Data	Central Line	Control Limits	
		Formula Using Factors in Table 26	Alternate Formula
No Standard Given			
Samples of equal size			
based on s	\bar{X}	$\bar{X} \pm E_3 \bar{s}$	$\bar{X} \pm 3\bar{s}/c_4$ (34)
based on R	\bar{X}	$\bar{X} \pm E_2 \bar{R}$	$\bar{X} \pm 3\bar{R}/d_2$ (35)
Samples of unequal size: σ computed from observed values of s per Section 3.9 or from observed values of R per Section 3.10(b)	\bar{X}		$\bar{X} \pm 3\hat{\sigma}$ (36) ^a
Standard Given			
Samples of equal or unequal size	μ_0		$\mu_0 \pm 3\sigma_0$ (37)
^a See Example 4 for determination of $\hat{\sigma}$ based on values of s and Example 6 for determination of $\hat{\sigma}$ based on values of R .			

^a See Example 4 for determination of $\hat{\sigma}$ based on values of s and Example 6 for determination of $\hat{\sigma}$ based on values of R .

3.30. CONTROL CHART FOR INDIVIDUALS, \bar{X} —USING MOVING RANGES

A. No Standard Given

Here the control chart lines are computed from the observed data. In this section, the symbol MR is used to signify the moving range. The control chart lines are as shown in Table 24 where

\bar{X} = the average of the individual observations,

MR = the mean moving range, (see Supplement 3.B, Note 7 for more general discussion) the average of the absolute values of successive differences between pairs of the individual observations, and

$n = 2$ for determining E_2 , D_3 , and D_4 .

See Example 24.

B. Standard Given

When μ_0 and σ_0 are given, the control chart lines are as shown in Table 25.

See Example 25.

EXAMPLES

3.31. ILLUSTRATIVE EXAMPLES—CONTROL, NO STANDARD GIVEN

Examples 1 to 11, inclusive, illustrate the use of the control chart method of analyzing data for control, when no standard is given (see Sections 3.7 to 3.17).

TABLE 25—Equations for Control Chart Lines

Chart For Individuals—Standard Given		
	Central Line	Control Limits
For individuals	μ_0	$\mu_0 \pm 3\sigma_0$ (40)
For moving ranges of two observations	$d_2\sigma_0$	$D_2\sigma_0 = 3.69\sigma_0$ $D_1\sigma_0 = 0$ (41)

Example 1: Control Charts for \bar{X} and s , Large Samples of Equal Size (Section 3.8A)

A manufacturer wished to determine if his product exhibited a state of control. In this case, the central lines and control limits were based solely on the data. Table 27 gives observed values of \bar{X} and s for daily samples of $n = 50$ observations each for ten consecutive days. Figure 2 gives the control charts for \bar{X} and s .

Central Lines

For \bar{X} : $\bar{\bar{X}} = 34.0$

For s : $\bar{s} = 4.40$

Control Limits

$n = 50$

For \bar{X} : $\bar{\bar{X}} \pm 3 \frac{\bar{s}}{\sqrt{n-0.5}} = 34.0 \pm 1.9,$
32.1 and 35.9

For s : $\bar{s} \pm 3 \frac{\bar{s}}{\sqrt{2n-2.5}} = 4.40 \pm 1.34,$
3.06 and 5.74

TABLE 24—Equations for Control Chart Lines

Chart for Individuals—Using Moving Ranges—No Standard Given		
	Central Line	Control Limits
For individuals	\bar{X}	$\bar{X} \pm E_2 MR = \bar{X} \pm 2.66 MR$ (38)
For moving ranges of two observations	\bar{R}	$D_4 MR = 3.27 MR$ $D_3 MR = 0$ (39)

TABLE 26—Factors for Computing Control Limits

Chart for Individuals—Associated with Chart for s or R Having Sample Size n									
Observations in Samples of Equal Size (from which \bar{s} or \bar{R} Has Been Determined)	2	3	4	5	6	7	8	9	10
Factors for control limits									
E_3	3.760	3.385	3.256	3.192	3.153	3.127	3.109	3.095	3.084
E_2	2.659	1.772	1.457	1.290	1.184	1.109	1.054	1.010	0.975

TABLE 27—Operating Characteristic, Daily Control Data

Sample	Sample Size, n	Average, \bar{X}	Standard Deviation, s
1	50	35.1	5.35
2	50	34.6	4.73
3	50	33.2	3.73
4	50	34.8	4.55
5	50	33.4	4.00
6	50	33.9	4.30
7	50	34.4	4.98
8	50	33.0	5.30
9	50	32.8	3.29
10	50	34.8	3.77
Total	500	340.0	44.00
Average	50	34.0	4.40

RESULTS

The charts give no evidence of lack of control. Compare with *Example 12*, in which the same data are used to test product for control at a specified level.

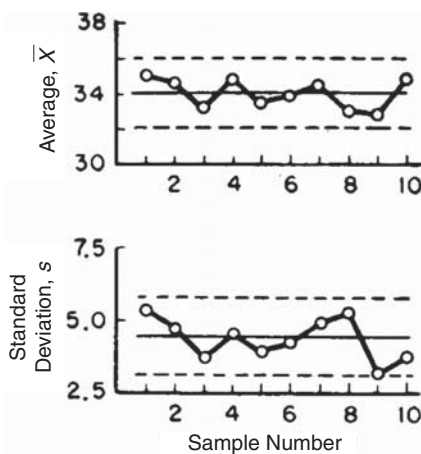


FIG. 2—Control charts for \bar{X} and s . Large samples of equal size, $n = 50$; no standard given.

Example 2: Control Charts for \bar{X} and s , Large Samples of Unequal Size (Section 3.8B)

To determine whether there existed any assignable causes of variation in quality for an important operating characteristic of a given product, the inspection results given in Table 28 were obtained from ten shipments whose samples were unequal in size; hence, control limits were computed separately for each sample size.

Figure 3 gives the control charts for \bar{X} and s .

Central Lines

For \bar{X} : $\bar{\bar{X}} = 53.8$

For s : $\bar{\bar{s}} = 3.39$

Control Limits

$$\text{For } \bar{X}: \bar{\bar{X}} \pm 3 \frac{\bar{s}}{\sqrt{n-0.5}} = 53.8 \pm \frac{10.17}{\sqrt{n-0.5}}$$

$$n = 25 : 51.7 \text{ and } 55.9$$

$$n = 50 : 52.4 \text{ and } 55.2$$

$$n = 100 : 52.8 \text{ and } 54.8$$

$$\text{For } s: \bar{\bar{s}} \pm 3 \frac{\bar{s}}{\sqrt{2n-2.5}} = 3.39 \pm \frac{10.17}{\sqrt{2n-2.5}}$$

$$n = 25 : 1.91 \text{ and } 4.87$$

$$n = 50 : 2.36 \text{ and } 4.42$$

$$n = 100 : 2.67 \text{ and } 4.11$$

RESULTS

Lack of control is indicated with respect to both \bar{X} and s . Corrective action is needed to reduce the variability between shipments.

Example 3: Control Charts for \bar{X} and s , Small Samples of Equal Size (Section 3.9A)

Table 29 gives the width in inches to the nearest 0.0001 in. measured prior to exposure for ten sets of corrosion specimens of Grade BB zinc. These two groups of five sets each were selected for illustrative purposes from a large number of sets of specimens consisting of six specimens each used in atmosphere exposure tests sponsored by ASTM. In each of the two groups, the five sets correspond to five different millings that were employed in the preparation of the specimens. Figure 4 shows control charts for \bar{X} and s .

RESULTS

The chart for averages indicates the presence of assignable causes of variation in width, \bar{X} from set to set, that is, from

TABLE 28—Operating Characteristic, Mechanical Part			
Shipment	Sample Size, n	Average, \bar{X}	Standard Deviation, s
1	50	55.7	4.35
2	50	54.6	4.03
3	100	52.6	2.43
4	25	55.0	3.56
5	25	53.4	3.10
6	50	55.2	3.30
7	100	53.3	4.18
8	50	52.3	4.30
9	50	53.7	2.09
10	50	54.3	2.67
Total	550	$\Sigma n \bar{X} = 29,590.0$	$\Sigma ns = 1,864.50$
Weighted average	55	53.8	3.39

milling to milling. The pattern of points for averages indicates a systematic pattern of width values for the five millings, a factor that required recognition in the analysis of the corrosion test results.

Central Lines

$$\text{For } \bar{X}: \bar{\bar{X}} = 0.49998$$

$$\text{For } s: \bar{s} = 0.00025$$

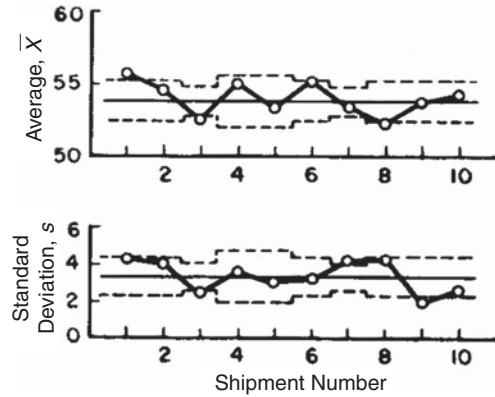


FIG. 3—Control charts for \bar{X} and s . Large samples of unequal size, $n = 25, 50, 100$; no standard given.

Control Limits

$$n = 6$$

$$\text{For } \bar{X}: \bar{\bar{X}} \pm A_3\bar{s} = 0.49998 \pm (1.287)(0.00025) \\ 0.49966 \text{ and } 0.50030$$

$$\text{For } s: B_4\bar{s} = (1.970)(0.00025) = 0.00049$$

$$B_3\bar{s} = (0.030)(0.00025) = 0.00001$$

Example 4: Control Charts for \bar{X} and s , Small Samples of Unequal Size (Section 3.9B)

Table 30 gives interlaboratory calibration check data on 21 horizontal tension testing machines. The data represent tests on No. 16 wire. The procedure is similar to that given in Example 3, but indicates a suggested method of computation when the samples are not equal in size. Figure 5 gives control charts for \bar{X} and s .

$$\hat{\sigma} = \frac{1}{21} \left(\frac{2.41}{0.9213} + \frac{15.34}{0.9400} \right) = 0.902$$

TABLE 29—Width in Inches, Specimens of Grade BB Zinc									
Set	Measured Values								
	X_1	X_2	X_3	X_4	X_5	X_6	Average, \bar{X}	Standard Deviation, s	Range, R
Group 1									
1	0.5005	0.5000	0.5008	0.5000	0.5005	0.5000	0.50030	0.00035	0.0008
2	0.4998	0.4997	0.4998	0.4994	0.4999	0.4998	0.49973	0.00018	0.0005
3	0.4995	0.4995	0.4995	0.4995	0.4995	0.4996	0.49952	0.00004	0.0001
4	0.4998	0.5005	0.5005	0.5002	0.5003	0.5004	0.50028	0.00026	0.0007
5	0.5000	0.5005	0.5008	0.5007	0.5008	0.5010	0.50063	0.00035	0.0010
Group 2									
6	0.5008	0.5009	0.5010	0.5005	0.5006	0.5009	0.50078	0.00019	0.0005
7	0.5000	0.5001	0.5002	0.4995	0.4996	0.4997	0.49985	0.00029	0.0007
8	0.4993	0.4994	0.4999	0.4996	0.4996	0.4997	0.49958	0.00021	0.0006
9	0.4995	0.4995	0.4997	0.4992	0.4995	0.4992	0.49943	0.00020	0.0005
10	0.4994	0.4998	0.5000	0.4990	0.5000	0.5000	0.49970	0.00041	0.0010
Average							0.49998	0.00025	0.00064

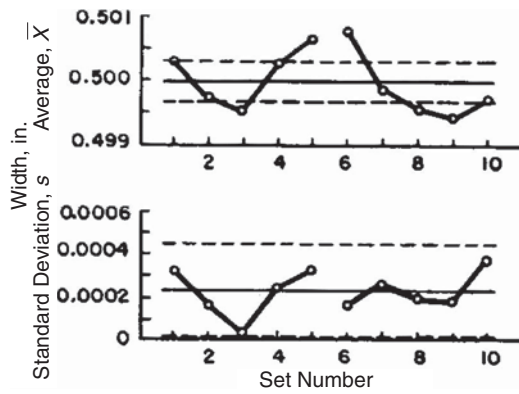


FIG. 4—Control chart for \bar{X} and s . Small samples of equal size, $n = 6$; no standard given.

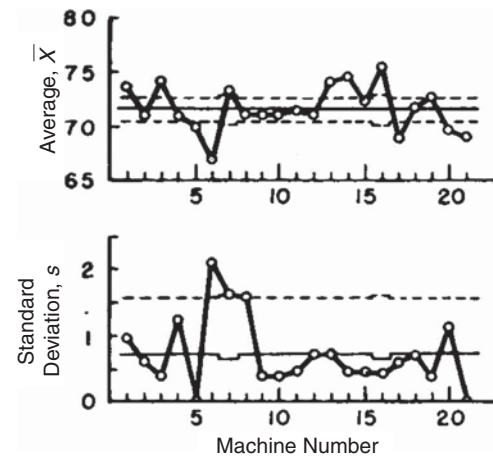


FIG. 5—Control chart for \bar{X} and s . Small samples of unequal size, $n = 4$; no standard given.

TABLE 30—Interlaboratory Calibration, Horizontal Tension Testing Machines

Machine	Number of Tests	Test Value					Average, \bar{X}	Standard Deviation, s		Range, R	
		1	2	3	4	5		$n = 4$	$n = 5$	$n = 4$	$n = 5$
1	5	73	73	73	75	75	73.8	...	1.10	...	2
2	5	70	71	71	71	72	71.0	...	0.71	...	2
3	5	74	74	74	74	75	74.2	...	0.45	...	1
4	5	70	70	70	72	73	71.0	...	1.41	...	3
5	5	70	70	70	70	70	70.0	...	0	...	0
6	5	65	65	66	69	70	67.0	...	2.35	...	5
7	4	72	72	74	76	...	73.5	1.91	...	4	...
8	5	69	70	71	73	73	71.2	...	1.79	...	4
9	5	71	71	71	71	72	71.2	...	0.45	...	1
10	5	71	71	71	71	72	71.2	...	0.45	...	1
11	5	71	71	72	72	72	71.6	...	0.55	...	1
12	5	70	71	71	72	72	71.2	...	0.55	...	2
13	5	73	74	74	75	75	74.2	...	0.84	...	2
14	5	74	74	75	75	75	74.6	...	0.55	...	1
15	5	72	72	72	73	73	72.4	...	0.55	...	1
16	4	75	75	75	76	...	75.3	0.50	...	1	...
17	5	68	69	69	69	70	69.0	...	0.71	...	2
18	5	71	71	72	72	73	71.8	...	0.84	...	2
19	5	72	73	73	73	73	72.8	...	0.45	...	1
20	5	68	69	70	71	71	69.8	...	1.30	...	3
21	5	69	69	69	69	69	69.0	...	0	...	0
Total	103		Weighted average $\bar{X} = 71.65$					2.41	15.34	5	34

Central Lines

For \bar{X} : $\bar{\bar{X}} = 71.65$

$$\text{For } s: n = 4: \bar{s} = c_4 \hat{\sigma} = (0.9213)(0.902) = 0.831$$

$$n = 5: \bar{s} = c_4 \hat{\sigma} = (0.9400)(0.902) = 0.848$$

Control Limits

For \bar{X} : $n = 4: \bar{\bar{X}} \pm A_3 \bar{s} =$

$$71.65 \pm (1.628)(0.831),$$

$$73.0, \text{ and } 70.3$$

 $n = 5: \bar{\bar{X}} \pm A_3 \bar{s} =$

$$71.65 \pm (1.427)(0.848),$$

$$72.9, \text{ and } 70.4$$

For s : $n = 4: B_4 \bar{s} = (2.266)(0.831) = 1.88$

$$B_3 \bar{s} = (0)(0.831) = 0$$

 $n = 5: B_4 \bar{s} = (2.089)(0.848) = 1.77$

$$B_3 \bar{s} = (0)(0.848) = 0$$

RESULTS

The calibration levels of machines were not controlled at a common level; the averages of six machines are above and the averages of five machines are below the control limits. Likewise, there is an indication that the variability *within* machines is not in statistical control, because three machines, Numbers 6, 7, and 8, have standard deviations outside the control limits.

Example 5: Control Charts for \bar{X} and R, Small Samples of Equal Size (Section 3.10A)

Same data as in Example 3, Table 29. Use is made of control charts for averages and ranges rather than for averages and standard deviations. Figure 6 shows control charts for \bar{X} and R.

RESULTS

The results are practically identical in all respects with those obtained by using averages and standard deviations, Fig. 4, Example 3.

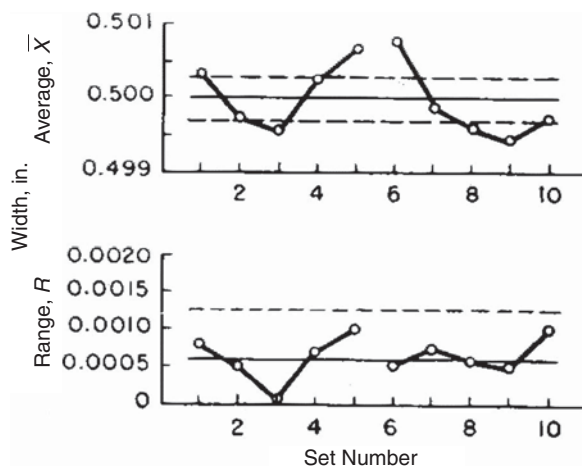


FIG. 6—Control charts for \bar{X} and R. Small samples of equal size, $n = 6$; no standard given.

Central Lines

For \bar{X} : $\bar{\bar{X}} = 0.49998$ For R: $\bar{R} = 0.00064$

Control Limits

 $n = 6$ For \bar{X} : $\bar{\bar{X}} \pm A_2 \bar{R}$

$$= 0.49998 \pm (0.483)(0.00064)$$

$$= 0.50029 \text{ and } 0.49967$$

For R: $D_4 \bar{R} = (2.004)(0.00064) = 0.00128$

$$D_3 \bar{R} = (0)(0.00064) = 0$$

Example 6: Control Charts for \bar{X} and R, Small Samples of Unequal Size (Section 3.10B)

Same data as in Example 4, Table 8. In the analysis and control charts, the range is used instead of the standard deviation. The procedure is similar to that given in Example 5, but indicates a suggested method of computation when samples are not equal in size. Figure 7 gives control charts for \bar{X} and R.

$\hat{\sigma}$ is determined from the tabulated ranges given in Example 4, using a similar procedure to that given in Example 4 for standard deviations where samples are not equal in size, that is

$$\hat{\sigma} = \frac{1}{21} \left(\frac{5}{2.059} + \frac{34}{2.326} \right) = 0.812$$

RESULTS

The results are practically identical in all respects with those obtained by using averages and standard deviations (Fig. 5, Example 4).

Central Lines

For \bar{X} : $\bar{\bar{X}} = 71.65$ For R: $n = 4: \bar{R} = d_2 \hat{\sigma} =$

$$(2.059)(0.812) = 1.67$$

 $n = 5: \bar{R} = d_2 \hat{\sigma} =$

$$(2.326)(0.812) = 1.89$$

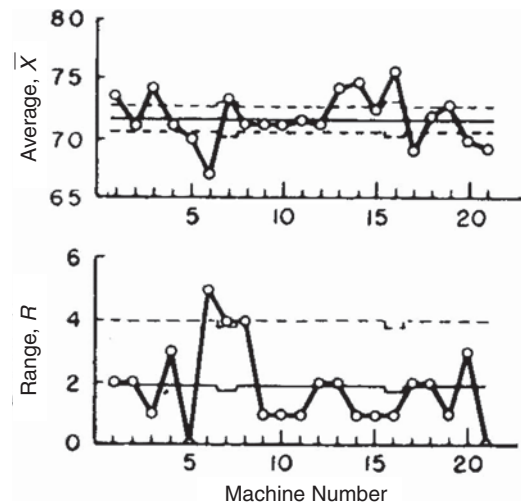


FIG. 7—Control charts for \bar{X} and R. Small samples of unequal size, $n = 4, 5$; no standard given.

Control Limits
 For \bar{X} : $n = 4$: $\bar{\bar{X}} \pm A_2\bar{R} =$
 $71.65 \pm (0.729)(1.67)$
 70.4 and 72.9

$n = 5$: $\bar{\bar{X}} \pm A_2\bar{R} =$
 $71.65 \pm (0.577)(1.89)$
 70.6 and 72.7

For R : $n = 4$: $D_4\bar{R} = (2.282)(1.67) = 3.8$
 $D_3\bar{R} = (0)(1.67) = 0$

$n = 5$: $D_4\bar{R} = (2.114)(1.89) = 4.0$
 $D_3\bar{R} = (0)(1.89) = 0$

Example 7: Control Charts for p , Samples of Equal Size (Section 3.13A) and np , Samples of Equal Size (Section 3.14)

Table 31 gives the number of nonconforming units found in inspecting a series of 15 consecutive lots of galvanized washers for finish nonconformities such as exposed steel, rough galvanizing. The lots were nearly the same size and a constant sample size of $n = 400$ was used. The fraction nonconforming for each sample was determined by dividing the number of nonconforming units found, np , by the sample size, n , and is listed in the table. Figure 8 gives the control chart for p , and Fig. 9 gives the control chart for np .

Note that these two charts are identical except for the vertical scale.

(A) CONTROL CHART FOR p

$$\begin{aligned}\text{Central Line} \\ \bar{p} &= \frac{33}{6,000} = 0.0055 \\ \bar{p} &= \frac{0.0825}{15} = 0.0055\end{aligned}$$

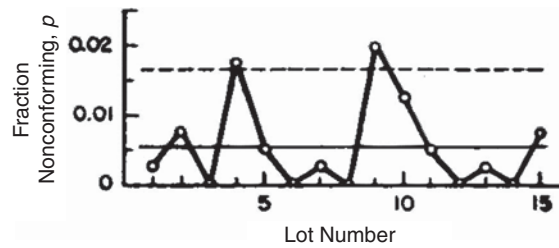


FIG. 8—Control chart for p . Samples of equal size, $n = 400$; no standard given.

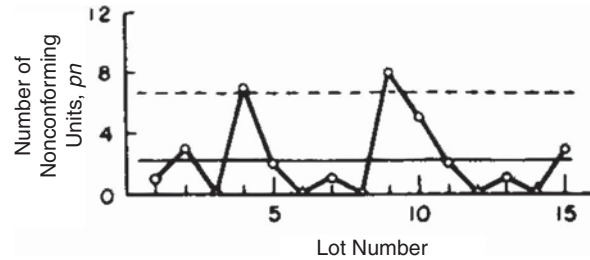


FIG. 9—Control chart for np . Samples of equal size, $n = 400$; no standard given.

$$\begin{aligned}\text{Control Limits} \\ n &= 400 \\ \bar{p} \pm 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} &= \\ 0.0055 \pm 3\sqrt{\frac{0.0055(0.9945)}{400}} &= \\ 0.0055 \pm 0.0111 &= \\ 0 \text{ and } 0.0166\end{aligned}$$

RESULTS

Lack of control is indicated; points for lots numbers 4 and 9 are outside the control limits.

TABLE 31—Finish Defects, Galvanized Washers

Lot	Sample Size, n	Number of Nonconforming Units, np	Fraction Nonconforming, p	Lot	Sample Size, n	Number of Nonconforming Units, np	Fraction Nonconforming, p
No. 1	400	1	0.0025	No. 9	400	8	0.0200
No. 2	400	3	0.0075	No. 10	400	5	0.0125
No. 3	400	0	0				
No. 4	400	7	0.0175	No. 11	400	2	0.0050
No. 5	400	2	0.0050	No. 12	400	0	0
				No. 13	400	1	0.0025
No. 6	400	0	0	No. 14	400	0	0
No. 7	400	1	0.0025	No. 15	400	3	0.0075
No. 8	400	0	0				
				Total	6,000	33	0.0825

(B) CONTROL CHART FOR np

Central Line
 $n = 400$
 $np = \frac{33}{15} = 2.2$

Control Limits
 $n = 400$
 $np \pm 3\sqrt{np} = 2.2 \pm 4.4$
0 and 6.6

Note

Because the value of np is 2.2, which is less than 4, the **NOTE** at the end of Section 3.13 (or 3.14) applies. The product of n and the upper control limit value for p is $400 \times 0.0166 = 6.64$. The nonintegral remainder, 0.64, is greater than one-half, and so the adjusted upper control limit value for p is $(6.64 + 1)/400 = 0.0191$. Therefore, only the point for Lot 9 is outside limits. For np , by the **NOTE** of Section 3.14, the adjusted upper control limit value is 7.6 with the same conclusion.

Example 8: Control Chart for p , Samples of Unequal Size (Section 3.13B)

Table 32 gives inspection results for surface defects on 31 lots of a certain type of galvanized hardware. The lot sizes

varied considerably and corresponding variations in sample sizes were used. Figure 10 gives the control chart for fraction nonconforming p . In practice, results are commonly expressed in “percent nonconforming,” using scale values of 100 times p .

Central Line
 $\bar{p} = \frac{268}{19\,510} = 0.01374$

Control Limits
 $\bar{p} \pm 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$

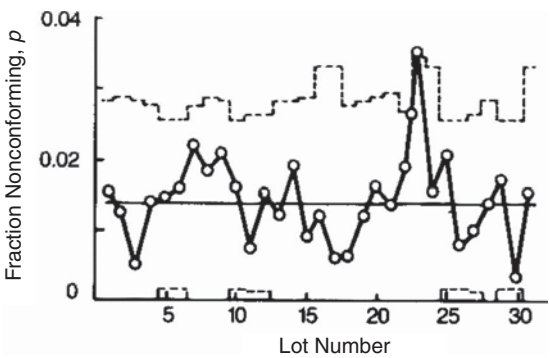


FIG. 10—Control chart for p . Samples of unequal size, $n = 200$ to 880; no standard given.

TABLE 32—Surface Defects, Galvanized Hardware							
Lot	Sample Size, n	Number of Nonconforming Units, np	Fraction Nonconforming, p	Lot	Sample Size, n	Number of Nonconforming Units, np	Fraction Nonconforming, p
No. 1	580	9	0.0155	No. 16	330	4	0.0121
No. 2	550	7	0.0127	No. 17	330	2	0.0061
No. 3	580	3	0.0052	No. 18	640	4	0.0063
No. 4	640	9	0.0141	No. 19	580	7	0.0121
No. 5	880	13	0.0148	No. 20	550	9	0.0164
No. 6	880	14	0.0159	No. 21	510	7	0.0137
No. 7	640	14	0.0219	No. 22	640	12	0.0188
No. 8	550	10	0.0182	No. 23	300	8	0.0267
No. 9	580	12	0.0207	No. 24	330	5	0.0152
No. 10	880	14	0.0159	No. 25	880	18	0.0205
No. 11	800	6	0.0075	No. 26	880	7	0.0080
No. 12	800	12	0.0150	No. 27	800	8	0.0100
No. 13	580	7	0.0121	No. 28	580	8	0.0138
No. 14	580	11	0.0190	No. 29	880	15	0.0170
No. 15	550	5	0.0091	No. 30	880	3	0.0034
				No. 31	330	5	0.0152
				Total	19,510	268	

$$\begin{aligned} &\text{For } n = 300 \\ &0.01374 \pm 3\sqrt{\frac{0.01374(0.98626)}{300}} = \\ &0.01374 \pm 3(0.006720) = \\ &0.01374 \pm 0.02016 \\ &0 \text{ and } 0.03390 \end{aligned}$$

$$\begin{aligned} &\text{For } n = 880 \\ &0.01374 \pm 3\sqrt{\frac{0.01374(0.98626)}{880}} = \\ &0.01374 \pm 3(0.003924) = \\ &0.01374 \pm 0.01177 \\ &0.00197 \text{ and } 0.02551 \end{aligned}$$

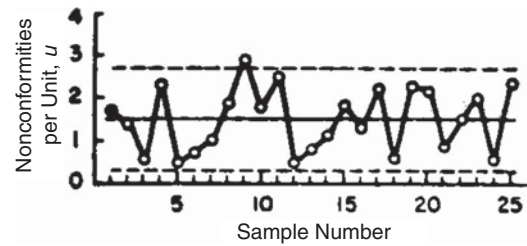


FIG. 11—Control chart for u . Samples of equal size, $n = 10$; no standard given.

found by the sample size and is listed in the table. Figure 11 gives the control chart for u , and Fig. 12 gives the control chart for c . Note that these two charts are identical except for the vertical scale.

RESULTS

A state of control may be assumed to exist since 25 consecutive subgroups fall within 3-sigma control limits. There are no points outside limits, so that the **NOTE** of Section 3.13 does not apply.

Example 9: Control Charts for u , Samples of Equal Size (Section 3.15A) and c , Samples of Equal Size (Section 3.16A)

Table 33 gives inspection results in terms of nonconformities observed in the inspection of 25 consecutive lots of burlap bags. Because the number of bags in each lot differed slightly, a constant sample size, $n = 10$ was used. All nonconformities were counted although two or more nonconformities of the same or different kinds occurred on the same bag. The nonconformities per unit value for each sample was determined by dividing the number of nonconformities

(a) u

Central Line

$$\bar{u} = \frac{37.5}{25} = 1.5$$

Control Limits

$$\begin{aligned} n &= 10 \\ \bar{u} \pm 3\sqrt{\frac{\bar{u}}{n}} &= \\ 1.50 \pm 3\sqrt{0.150} &= \\ 1.50 \pm 1.16 & \\ 0.34 \text{ and } 2.66 & \end{aligned}$$

(b) c

Central Line

$$\bar{c} = \frac{375}{25} = 15.0$$

TABLE 33—Number of Nonconformities in Consecutive Samples of Ten Units Each—Burlap Bags

Sample	Total Nonconformities in Sample c	Nonconformities per Unit u	Sample	Total Nonconformities in Sample, c	Nonconformities per Unit, u
1	17	1.7	13	8	0.8
2	14	1.4	14	11	1.1
3	6	0.6	15	18	1.8
4	23	2.3	16	13	1.3
5	5	0.5	17	22	2.2
6	7	0.7	18	6	0.6
7	10	1.0	19	23	2.3
8	19	1.9	20	22	2.2
9	29	2.9	21	9	0.9
10	18	1.8	22	15	1.5
11	25	2.5	23	20	2.0
12	5	0.5	24	6	0.6
			25	24	2.4
			Total	375	37.5

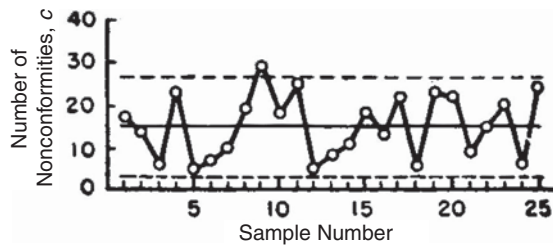


FIG. 12—Control chart for c . Samples of equal size, $n = 10$; no standard given.

$$\begin{aligned} \text{Control Limits} \\ n = 10 \\ \bar{c} \pm 3\sqrt{\bar{c}} = \\ 15.0 \pm 3\sqrt{15} = \\ 15.0 \pm 11.6 \\ 3.4 \text{ and } 26.6 \end{aligned}$$

RESULTS

Presence of assignable causes of variation is indicated by Sample 9. Because the value of nu is 15 (greater than 4), the NOTE at the end of Section 3.15 (or 3.16) does not apply.

Example 10: Control Chart for u , Samples of Unequal Size (Section 3.15B)

Table 34 gives inspection results for 20 lots of different sizes for which three different sample sizes were used, 20, 25, and 40. The observed nonconformities in this inspection cover all of the specified characteristics of a complex machine (Type A), including a large number of dimensional, operational, as well as physical and finish requirements. Because of the large number of tests and measurements required as well as possible occurrences of any minor observed irregularities, the expectancy of nonconformities per unit is high, although the majority of such nonconformities are of minor seriousness.

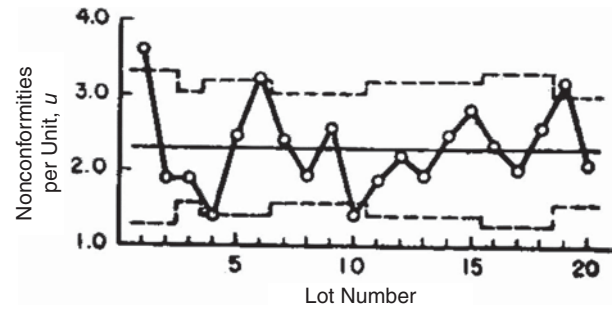


FIG. 13—Control chart for u . Samples of unequal size, $n = 20, 25, 40$; no standard given.

The nonconformities per unit value for each sample, number of nonconformities in sample divided by number of units in sample, was determined and these values are listed in the last column of the table. Figure 13 gives the control chart for u with control limits corresponding to the three different sample sizes.

$$\begin{aligned} \text{Central Line} \\ \bar{u} = \frac{1334}{580} = 2.30 \end{aligned}$$

$$\begin{aligned} \text{Control Limits} \\ n = 20 \\ \bar{u} \pm 3\sqrt{\frac{\bar{u}}{n}} = 2.30 \pm 1.02, \\ 1.28 \text{ and } 3.32 \\ n = 25 \\ \bar{u} \pm 3\sqrt{\frac{\bar{u}}{n}} = 2.30 \pm 0.91, \\ 1.39 \text{ and } 3.21 \\ n = 40 \\ \bar{u} \pm 3\sqrt{\frac{\bar{u}}{n}} = 2.30 \pm 0.72, \\ 1.58 \text{ and } 3.02 \end{aligned}$$

TABLE 34—Number of Nonconformities in Samples from 20 Successive Lots of Type A Machines

Lot	Sample Size, n	Total Nonconformities Sample, c	Nonconformities per Unit, u	Lot	Sample Size, n	Total Nonconformities Sample, c	Nonconformities per Unit, u
No. 1	20	72	3.60	No. 11	25	47	1.88
No. 2	20	38	1.90	No. 12	25	55	2.20
No. 3	40	76	1.90	No. 13	25	49	1.96
No. 4	25	35	1.40	No. 14	25	62	2.48
No. 5	25	62	2.48	No. 15	25	71	2.84
No. 6	25	81	3.24	No. 16	20	47	2.35
No. 7	40	97	2.42	No. 17	20	41	2.05
No. 8	40	78	1.95	No. 18	20	52	2.60
No. 9	40	103	2.58	No. 19	40	128	3.20
No. 10	40	56	1.40	No. 20	40	84	2.10
				Total	580	1,334	

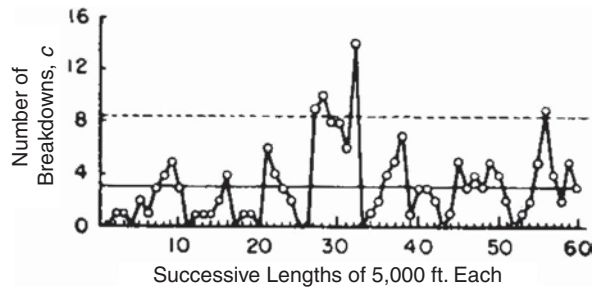


FIG. 14—Control chart for c . Samples of equal size, $n = 1$ standard length of 5,000 ft; no standard given.

(A) LENGTHS OF 5,000 FT EACH

$$\text{Central Line} \\ \bar{c} = \frac{187}{60} = 3.12$$

$$\text{Control Limits} \\ \bar{c} \pm 3\sqrt{\bar{c}} = \\ 6.23 \pm 3\sqrt{6.23} \\ 0 \text{ and } 13.72$$

(A) RESULTS

Presence of assignable causes of variation is indicated by length numbers 27, 28, 32, and 56 falling above the upper control limit. Because the value of $\bar{c} = n\bar{u}$ is 3.12 (less than 4), the **NOTE** at the end of Section 3.16 does apply. The non-integral remainder of the upper control limit value is 0.42. The upper control limit stands, as do the indications of lack of control.

(B) LENGTHS OF 10,000 FT EACH

$$\text{Central Line} \\ \bar{c} = \frac{187}{30} = 6.23$$

$$\text{Control Limits} \\ \bar{c} \pm 3\sqrt{\bar{c}} = \\ 6.23 \pm 3\sqrt{6.23} \\ 0 \text{ and } 13.72$$

(B) RESULTS

Presence of assignable causes of variation is indicated by length numbers 14, 15, 16, and 28 falling above the upper

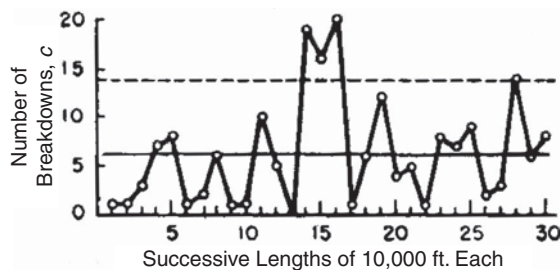


FIG. 15—Control chart for c . Samples of equal size, $n = 1$ standard length of 10,000 ft; no standard given.

control limit. Because the value of \bar{c} is 6.23 (greater than 4), the **NOTE** at the end of Section 3.16 does not apply.

3.32. ILLUSTRATIVE EXAMPLES—CONTROL WITH RESPECT TO A GIVEN STANDARD

Examples 12 to 21, inclusive, illustrate the use of the control chart method of analyzing data for control with respect to a given standard (see Sections 3.18 to 3.27).

Example 12: Control Charts for \bar{X} and s , Large Samples of Equal Size (Section 3.19)

A manufacturer attempted to maintain an aimed-at distribution of quality for a certain operating characteristic. The objective standard distribution which served as a target was defined by standard values: $\mu_0 = 35.00$ lb, and $\sigma_0 = 4.20$ lb. Table 36 gives observed values of \bar{X} and s for daily samples of $n = 50$ observations each for ten consecutive days. These data are the same as used in Example 1 and presented as Table 27. Figure 16 gives control charts for \bar{X} and s .

$$\text{Central Lines} \\ \text{For } \bar{X} : \mu_0 = 35.00 \\ \text{For } s : \sigma_0 = 4.20$$

$$\text{Control Limits} \\ n = 50 \\ \text{For } \bar{X} : \mu_0 \pm 3 \frac{\sigma_0}{\sqrt{n}} = 35.00 \pm 1.8, 33.2 \text{ and } 36.8 \\ \text{For } s : \left(\frac{4n-4}{4n-3} \right) \sigma_0 \pm 3 \frac{\sigma_0}{\sqrt{2n-1.5}} = 4.18 \pm 1.27, 2.19 \text{ and } 5.45$$

RESULTS

Lack of control at standard level is indicated on the eighth and ninth days. Compare with Example 1 in which the same data were analyzed for control without specifying a standard level of quality.

TABLE 36—Operating Characteristic, Daily Control Data

Sample	Sample Size, n	Average, \bar{X}	Standard Deviation, s
1	50	35.1	5.35
2	50	34.6	4.73
3	50	33.2	3.73
4	50	34.8	4.55
5	50	33.4	4.00
6	50	33.9	4.30
7	50	34.4	4.98
8	50	33.0	5.30
9	50	32.8	3.29
10	50	34.8	3.77

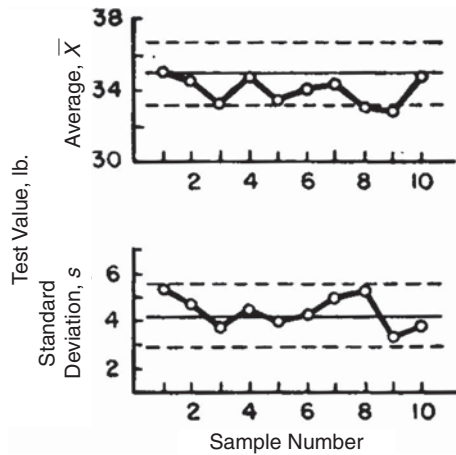


FIG. 16—Control charts for \bar{X} and s . Large samples of equal size, $n = 50$; μ_0, σ_0 given.

Example 13: Control Charts for \bar{X} and s , Large Samples of Unequal Size (Section 3.19)

For a product, it was desired to control a certain critical dimension, the diameter, with respect to day-to-day variation. Daily sample sizes of 30, 50, or 75 were selected and measured, the number taken depending on the quantity produced per day. The desired level was $\mu_0 = 0.20000$ in. with $\sigma_0 = 0.00300$ in. Table 37 gives observed values of \bar{X} and s for the samples from ten successive days' production. Figure 17 gives the control charts for \bar{X} and s .

Central Lines

For \bar{X} : $\mu_0 = 0.20000$

For s : $\sigma_0 = 0.00300$

Control Limits

For \bar{X} : $\mu_0 \pm 3 \frac{\sigma_0}{\sqrt{n}}$

$n = 30$

$0.2000 \pm 3 \frac{0.00300}{\sqrt{30}} =$
 0.20000 ± 0.00164
 0.19836 and 0.20164

$n = 50$

0.19873 and 0.20127

$n = 75$

0.19896 and 0.20104

For s : $c_4 \sigma_0 \pm 3 \frac{\sigma_0}{\sqrt{2n-1.5}}$

$n = 30$

$\left(\frac{116}{117}\right) 0.00300 \pm 3 \frac{0.00300}{\sqrt{58.5}} =$
 0.00297 ± 0.00118
 0.00180 and 0.00415

$n = 50$

0.00389 and 0.00208

$n = 75$

0.00225 and 0.00373

RESULTS

The charts give no evidence of significant deviations from standard values.

TABLE 37—Diameter in inches, Control Data

Sample	Sample Size, n	Average, \bar{X}	Standard Deviation, s
1	30	0.20133	0.00330
2	50	0.19886	0.00292
3	50	0.20037	0.00326
4	30	0.19965	0.00358
5	75	0.19923	0.00313
6	75	0.19934	0.00306
7	75	0.19984	0.00299
8	50	0.19974	0.00335
9	50	0.20095	0.00221
10	30	0.19937	0.00397

Example 14: Control Chart for \bar{X} and s , Small Samples of Equal Size (Section 3.19)

Same product and characteristic as in Example 13, but in this case it is desired to control the diameter of this product with respect to sample variations during each day, because samples of ten were taken at definite intervals each day. The desired level is $\mu_0 = 0.20000$ in. with $\sigma_0 = 0.00300$ in. Table 38 gives observed values of \bar{X} and s for ten samples of ten each taken during a single day. Figure 18 gives the control charts for \bar{X} and s .

Central Lines

For \bar{X} : $\mu_0 = 0.20000$

$n = 10$

For s : $c_4 \sigma_0 = (0.9727)(0.00300) = 0.00292$

Control Limits

$n = 10$

For \bar{X} : $\mu_0 \pm A \sigma_0 =$
 $0.20000 \pm (0.949)(0.00300),$
 0.19715 and 0.20285

For s : $B_6 \sigma_0 = (1.669)(0.00300) = 0.00501$

$B_5 \sigma_0 = (0.276)(0.00300) = 0.00083$

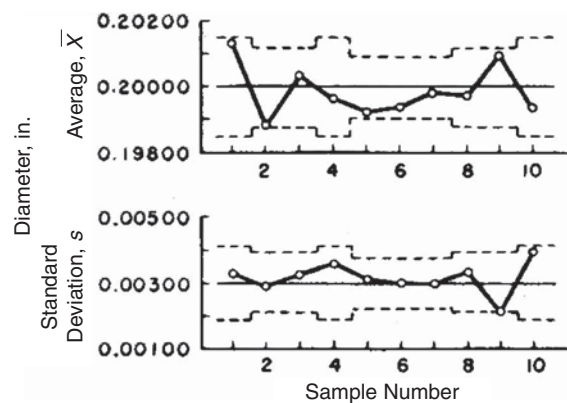
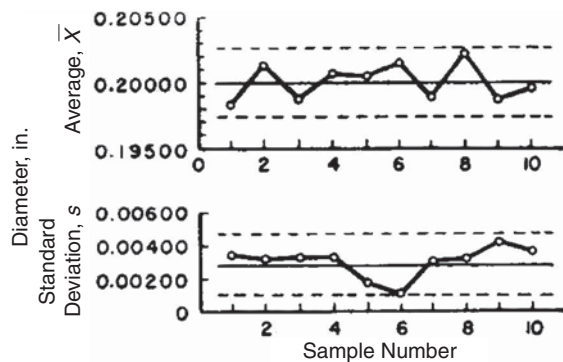


FIG. 17—Control charts for \bar{X} and s . Large samples of unequal size, $n = 30, 50, 75$; μ_0, σ_0 given.

TABLE 38—Control Data for One Day's Product

Sample	Sample Size, n	Average, \bar{X}	Standard Deviation, s
1	10	0.19838	0.00350
2	10	0.20126	0.00304
3	10	0.19868	0.00333
4	10	0.20071	0.00337
5	10	0.20050	0.00159
6	10	0.20137	0.00104
7	10	0.19883	0.00299
8	10	0.20218	0.00327
9	10	0.19868	0.00431
10	10	0.19968	0.00356

**FIG. 18—Control charts for \bar{X} and s . Small samples of equal size, $n = 10$; μ_0, σ_0 given.****RESULTS**

No lack of control indicated.

Example 15: Control Chart for \bar{X} and s , Small Samples of Unequal Size (Section 3.19)

A manufacturer wished to control the resistance of a certain product after it had been operating for 100 h, where $\mu_0 = 150 \Omega$ and $\sigma_0 = 7.5 \Omega$ from each of 15 consecutive lots, he selected a random sample of five units and subjected them to the operating test for 100 h. Due to mechanical failures, some of the units in the sample failed before the completion of 100 h of operation. Table 39 gives the averages and standard deviations for the 15 samples together with their sample sizes. Figure 19 gives the control charts for \bar{X} and s .

Central Lines

For \bar{X} : $\mu_0 = 150$

$$n = 3$$

$$\mu_0 \pm A\sigma_0 = 150 \pm 1.732(7.5)$$

$$137.0 \text{ and } 163.0$$

$$n = 4$$

$$\mu_0 \pm A\sigma_0 = 150 \pm 1.500(7.5)$$

$$138.8 \text{ and } 161.2$$

$$n = 5$$

$$\mu_0 \pm A\sigma_0 = 150 \pm 1.342(7.5)$$

$$139.9 \text{ and } 160.1$$

For s : $\sigma_0 = 7.5$

$$n = 3$$

$$c_4\sigma_0 = (0.8862)(7.5) = 6.65$$

$$n = 4$$

$$c_4\sigma_0 = (0.9213)(7.5) = 6.91$$

$$n = 5$$

$$c_4\sigma_0 = (0.9400)(7.5) = 7.05$$

For s : $\sigma_0 = 7.5$

$$n = 3: B_6\sigma_0 = (2.276)(7.5) = 17.07$$

$$B_5\sigma_0 = (0)(7.5) = 0$$

$$n = 4: B_6\sigma_0 = (2.088)(7.5) = 15.66$$

$$B_5\sigma_0 = (0)(7.5) = 0$$

$$n = 5: B_6\sigma_0 = (1.964)(7.5) = 14.73$$

$$B_5\sigma_0 = (0)(7.5) = 0$$

TABLE 39—Resistance in ohms after 100-h Operation, Lot-by-Lot Control Data

Sample	Sample Size, n	Average, \bar{X}	Standard Deviation, s	Sample	Sample Size, n	Average, \bar{X}	Standard Deviation, s
1	5	154.6	12.20	9	5	156.2	8.92
2	5	143.4	9.75	10	4	137.5	3.24
3	4	160.8	11.20	11	5	153.8	6.85
4	3	152.7	7.43	12	5	143.4	7.64
5	5	136.0	4.32	13	4	156.0	10.18
6	3	147.3	8.65	14	5	149.8	8.86
7	3	161.7	9.23	15	3	138.2	7.38
8	5	151.0	7.24				

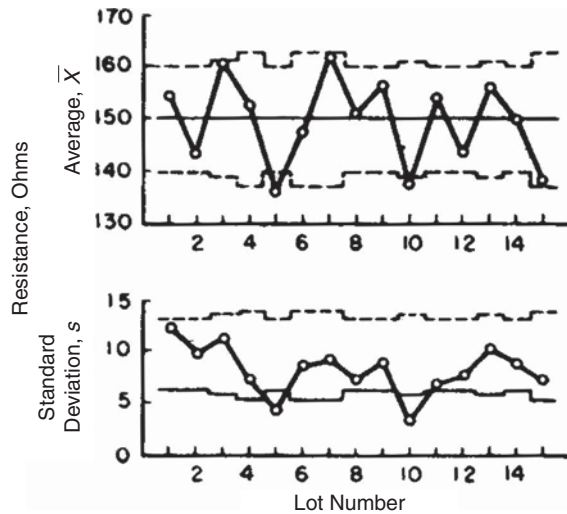


FIG. 19—Control charts for \bar{X} and s . Small samples of unequal size, $n = 3, 4, 5$; μ_0, σ_0 given.

RESULTS

Evidence of lack of control is indicated because samples from lots Numbers 5 and 10 have averages below their lower control limit. No standard deviation values are outside their control limits. Corrective action is required to reduce the variation between lot averages.

Example 16: Control Charts for \bar{X} and R , Small Samples of Equal Size (Sections 3.19 and 3.20)

Consider the same problem as in *Example 12* where $\mu_0 = 35.00$ lb and $\sigma_0 = 4.20$ lb. The manufacturer wished to control variations in quality from lot to lot by taking a small sample from each lot. Table 40 gives observed values of \bar{X} and R for samples of $n = 5$ each, selected from ten consecutive lots. Because the sample size n is less than ten, actually five, he elected to use control charts for \bar{X} and R rather than for \bar{X} and s . Figure 20 gives the control charts for \bar{X} and R .

Lot	Sample Size, n	Average, \bar{X}	Range, R
No. 1	5	36.0	6.6
No. 2	5	31.4	0.5
No. 3	5	39.0	15.1
No. 4	5	35.6	8.8
No. 5	5	38.8	2.2
No. 6	5	41.6	3.5
No. 7	5	36.2	9.6
No. 8	5	38.0	9.0
No. 9	5	31.4	20.6
No. 10	5	29.2	21.7

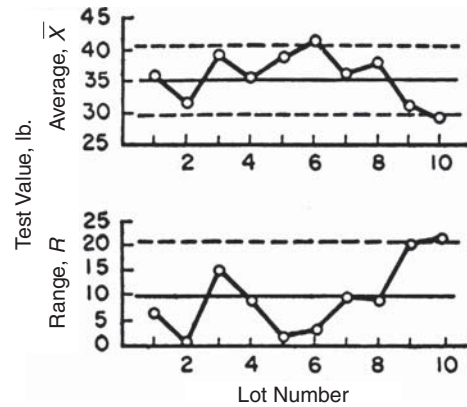


FIG. 20—Control charts for \bar{X} and R . Small samples of equal size, $n = 5$; μ_0, σ_0 given.

Central Lines

For \bar{X} : $\mu_0 = 35.00$

$n = 5$

For R : $d_2\sigma_0 = 2.326(4.20) = 9.8$

Control Limits

$n = 5$

For \bar{X} : $\mu_0 \pm A\sigma_0 =$
 $35.00 \pm (1.342)(4.20)$
 29.4 and 40.6

For R : $d_2\sigma_0 = (4.918)(4.20) = 20.7$

$A_1\sigma_0 = (0)(4.20) = 0$

RESULTS

Lack of control at the standard level is indicated by results for lot numbers 6 and 10. Corrective action is required both with respect to averages and with respect to variability within a lot.

Example 17: Control Charts for p , Samples of Equal Size (Section 3.23) and np , Samples of Equal Size (Section 3.24)

Consider the same data as in *Example 7*, Table 31. The manufacturer wishes to control his process with respect to finish on galvanized washers at a level such that the fraction nonconforming $p_0 = 0.0040$ (4 nonconforming washers per 1,000). Table 31 of *Example 7* gives observed values of “number of nonconforming units” for finish nonconformities such as exposed steel, rough galvanizing in samples of 400 washers drawn from 15 successive lots. Figure 21 shows the control chart for p , and Fig. 22 gives the control chart for np . In practice, only one of these control charts would be used because, except for change of scale, the two charts are identical.

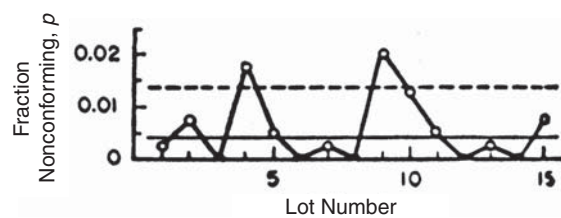


FIG. 21—Control chart for p . Samples of equal size, $n = 400$; p_0 given.

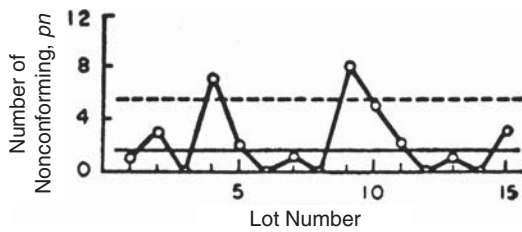


FIG. 22—Control chart for np . Samples of equal size, $n = 400$; p_0 given.

(A) p

Central Line
 $p_0 = 0.0040$

$$\begin{aligned} \text{Control Limits} \\ n = 400 \\ p_0 \pm 3\sqrt{\frac{p_0(1-p_0)}{n}} = \\ 0.0040 \pm 3\sqrt{\frac{0.0040(0.9960)}{400}} = \\ 0.0040 \pm 0.0095 \\ 0 \text{ and } 0.0135 \end{aligned}$$

(B) np

Central Line
 $np_0 = 0.0040(400) = 1.6$

Control Limits

EXTRACT FORMULA:

$$\begin{aligned} n = 400 \\ np_0 \pm 3\sqrt{np_0(1-p_0)} = \\ 1.6 \pm 3\sqrt{1.6(0.996)} = \\ 1.6 \pm 3\sqrt{1.5936} = \\ 1.6 \pm 3(1.262) \\ 0 \text{ and } 5.4 \end{aligned}$$

SIMPLIFIED APPROXIMATE FORMULA:

$n = 400$

Because p_0 is small, replace Eq 29 by Eq 30

$$\begin{aligned} np_0 \pm 3\sqrt{np_0} = \\ 1.6 \pm 3\sqrt{1.6} = \\ 1.6 \pm 3(1.265) \\ 0 \text{ and } 5.4 \end{aligned}$$

RESULTS

Lack of control of quality is indicated with respect to the desired level; lot numbers 4 and 9 are outside control limits.

Note

Because the value of np_0 is 1.6, less than 4, the **NOTE** at the end of Section 3.13 (or 3.14) applies, as mentioned at the end of Section 3.23 (or 3.24). The product of n and the upper control limit value for p is $400 \times 0.0135 = 5.4$. The nonintegral remainder, 0.4, is less than one-half. The upper control limit stands as does the indication of lack of control

to p_0 . For np , by the **NOTE** of Section 3.14, the same conclusion follows.

Example 18: Control Chart for p (Fraction Nonconforming), Samples of Unequal Size (Section 3.23e)

The manufacturer wished to control the quality of a type of electrical apparatus with respect to two adjustment characteristics at a level such that the fraction nonconforming $p_0 = 0.0020$ (2 nonconforming units per 1,000). Table 41 gives observed values of "number of nonconforming units" for this item found in samples drawn from successive lots.

Sample sizes vary considerably from lot to lot and, hence, control limits are computed for each sample. Equivalent control limits for "number of nonconforming units," np , are shown in column 5 of the table. In this way, the original records showing "number of nonconforming units" may be compared directly with control limits for np . Figure 23 shows the control chart for p .

Central Line for p
 $p_0 = 0.0020$

$$\text{Control Limits for } p \\ p_0 \pm 3\sqrt{\frac{p_0(1-p_0)}{n}}$$

For $n = 600$:

$$\begin{aligned} 0.0020 \pm 3\sqrt{\frac{0.002(0.998)}{600}} = \\ 0.0020 \pm 3(0.001824) \\ 0 \text{ and } 0.0075 \end{aligned}$$

(same procedure for other values of n)

Control Limits for np
Using Eq 3.30 for np ,
 $np_0 \pm 3\sqrt{np_0}$

For $n = 600$:

$$1.2 \pm 3\sqrt{1.2} = 1.2 \pm 3(1.095), \\ 0 \text{ and } 4.5$$

(same procedure for other values of n)

RESULTS

Lack of control and need for corrective action indicated by results for lots numbers 10 and 19.

Note

The values of np_0 for these lots are 4.0 and 2.6, respectively. The **NOTE** at the end of Section 3.13 (or 3.14) applies to lot number 19. The product of n and the upper control limit value for p is $1,300 \times 0.0057 = 7.41$. The nonintegral remainder is 0.41, less than one-half. The upper control limit stands, as does the indication of lack of control at p_0 . For np , by the **NOTE** of Section 3.14, the same conclusion follows.

Example 19: Control Chart for p (Fraction Rejected), Total and Components, Samples of Unequal Size (Section 3.23)

A control device was given a 100% inspection in lots varying in size from about 1,800 to 5,000 units, each unit being tested and inspected with respect to 23 *essentially independent*

TABLE 41—Adjustment Irregularities, Electrical Apparatus

Lot	Sample Size, n	Number of Nonconforming Units	Fraction Nonconforming, p	Upper Control Limit for np	Upper Control Limit for p
No. 1	600	2	0.0033	4.5	0.0075
No. 2	1,300	2	0.0015	7.4	0.0057
No. 3	2,000	1	0.0005	10.0	0.0050
No. 4	2,500	1	0.0004	11.7	0.0047
No. 5	1,550	5	0.0032	8.4	0.0054
No. 6	2,000	2	0.0010	10.0	0.0050
No. 7	1,550	0	0.0000	8.4	0.0054
No. 8	780	3	0.0038	5.3	0.0068
No. 9	260	0	0.0000	2.7	0.0103
No. 10	2,000	15	0.0075	10.0	0.0050
No. 11	1,550	7	0.0045	8.4	0.0054
No. 12	950	2	0.0021	6.0	0.0063
No. 13	950	5	0.0053	6.0	0.0063
No. 14	950	2	0.0021	6.0	0.0063
No. 15	35	0	0.0000	0.9	0.0247
No. 16	330	3	0.0091	3.1	0.0094
No. 17	200	0	0.0000	2.3	0.0115
No. 18	600	4	0.0067	4.5	0.0075
No. 19	1,300	8	0.0062	7.4	0.0057
No. 20	780	4	0.0051	5.3	0.0068

characteristics. These 23 characteristics were grouped into three groups designated Groups A, B, and C, corresponding to three successive inspections.

A unit found nonconforming at any time with respect to any one characteristic was immediately rejected; hence units found nonconforming in, say, the Group A inspection were not subjected to the two subsequent group inspections. In fact, the number of units inspected for each characteristic in a group itself will differ from characteristic to characteristic if nonconformities with respect to the characteristics in a group occur, the last characteristic in the group having the smallest sample size.

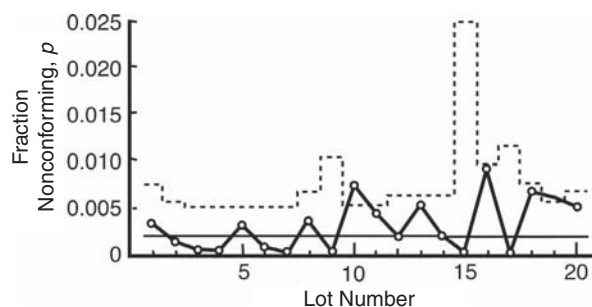


FIG. 23—Control chart for p . Samples of unequal size, to 2,500; p_0 given.

Because 100% inspection is used, no additional units are available for inspection to maintain a constant sample size for all characteristics in a group or for all the component groups. The fraction nonconforming with respect to each characteristic is sufficiently small so that the error within a group, although rather large between the first and last characteristic inspected by one inspection group, can be neglected for practical purposes. Under these circumstances, the number inspected for any group was equal to the lot size diminished by the number of units rejected in the preceding inspections.

Part 1 of Table 42 gives the data for twelve successive lots of product, and shows for each lot inspected the total fraction rejected as well as the number and fraction rejected at each inspection station. Part 2 of Table 42 gives values of p_0 , fraction rejected, at which levels the manufacturer desires to control this device, with respect to all 23 characteristics combined and with respect to the characteristics tested and inspected at each of the three inspection stations. Note that the p_0 for all characteristics (in terms of nonconforming units) is less than the sum of the p_0 values for the three component groups, because nonconformities from more than one characteristic or group of characteristics may occur on a single unit. Control limits, lower and upper, in terms of fraction rejected are listed for each lot size using the initial lot size as the sample size for all characteristics combined and the lot

TABLE 42—Inspection Data for 100% Inspection—Control Device

Observed Number of Rejects and Fraction Rejected												
All Groups Combined				Group A			Group B			Group C		
Lot	Lot Size, n	Total Rejected		Lot Size, n	Rejected		Lot Size, n	Rejected		Lot Size, n	Rejected	
		Number	Fraction		Number	Fraction		Number	Fraction		Number	Fraction
No. 1	4,814	914	0.190	4,814	311	0.065	4,503	253	0.056	4,250	350	0.082
No. 2	2,159	359	0.166	2,159	128	0.059	2,031	105	0.052	1,926	126	0.065
No. 3	3,089	565	0.183	3,089	195	0.063	2,894	149	0.051	2,745	221	0.081
No. 4	3,156	626	0.198	3,156	233	0.074	2,923	142	0.049	2,781	251	0.090
No. 5	2,139	434	0.203	2,139	146	0.068	1,993	101	0.051	1,892	187	0.099
No. 6	2,588	503	0.194	2,588	177	0.068	2,411	151	0.063	2,260	175	0.077
No. 7	2,510	487	0.194	2,510	143	0.057	2,367	116	0.049	2,251	228	0.101
No. 8	4,103	803	0.196	4,103	318	0.078	3,785	242	0.064	3,543	243	0.069
No. 9	2,992	547	0.183	2,992	208	0.070	2,784	130	0.047	2,654	209	0.079
No. 10	3,545	643	0.181	3,545	172	0.049	3,373	180	0.053	3,193	291	0.091
No. 11	1,841	353	0.192	1,841	97	0.053	1,744	119	0.068	1,625	137	0.084
No. 12	2,748	418	0.152	2,748	141	0.051	2,607	114	0.044	2,493	163	0.065

Central Lines and Control Limits, Based on Standard p_0 Values

	All Groups Combined	Group A	Group B	Group C
Central Lines				
$p_0 =$	0.180	0.070	0.050	0.080
Lot	Control Limits			
No. 1	0.197 and 0.163	0.081 and 0.059	0.060 and 0.040	0.093 and 0.067
No. 2	0.205 and 0.155	0.086 and 0.054	0.064 and 0.036	0.099 and 0.061
No. 3	0.201 and 0.159	0.084 and 0.056	0.062 and 0.038	0.096 and 0.064
No. 4	0.200 and 0.160	0.084 and 0.056	0.062 and 0.038	0.095 and 0.065
No. 5	0.205 and 0.155	0.086 and 0.054	0.065 and 0.035	0.099 and 0.061
No. 6	0.203 and 0.157	0.085 and 0.055	0.063 and 0.037	0.097 and 0.063
No. 7	0.203 and 0.157	0.085 and 0.055	0.064 and 0.036	0.097 and 0.063
No. 8	0.198 and 0.162	0.082 and 0.058	0.061 and 0.039	0.094 and 0.066
No. 9	0.201 and 0.159	0.084 and 0.056	0.062 and 0.038	0.096 and 0.064
No. 10	0.200 and 0.160	0.083 and 0.057	0.061 and 0.039	0.094 and 0.066
No. 11	0.207 and 0.153	0.088 and 0.052	0.066 and 0.034	0.100 and 0.060
No. 12	0.202 and 0.158	0.085 and 0.055	0.063 and 0.037	0.096 and 0.064

size available at the beginning of inspection and test for each group as the sample size for that group.

Figure 24 shows four control charts, one covering all rejections combined for the control device and three other charts covering the rejections for each of the three inspection stations for Group A, Group B, and Group C characteristics, respectively. Detailed computations for the overall

results for one lot and one of its component groups are given.

Central Lines
See Table 42

Control Limits
See Table 42

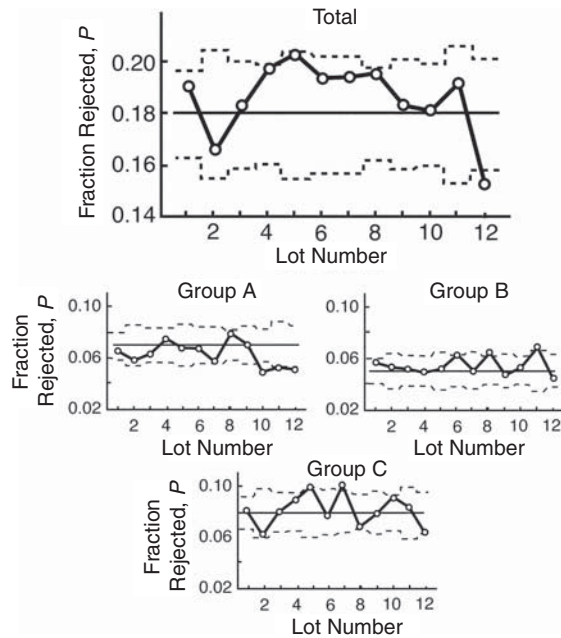


FIG. 24—Control charts for p (fraction rejected) for total and components. Samples of unequal size, $n = 1,625$ to $4,814$; p_0 given.

$$\begin{aligned}
 &\text{For Lot Number 1} \\
 &\quad \text{Total: } n = 4814 \\
 &\quad p_0 \pm 3\sqrt{\frac{p_0(1-p_0)}{n}} = \\
 &\quad 0.180 \pm 3\sqrt{\frac{0.180(0.820)}{4814}} = \\
 &\quad 0.180 \pm 3(0.0055) \\
 &\quad 0.163 \text{ and } 0.197 \\
 &\quad \text{Group C: } n = 4250 \\
 &\quad p_0 \pm 3\sqrt{\frac{p_0(1-p_0)}{n}} = \\
 &\quad 0.080 \pm 3\sqrt{\frac{0.080(0.920)}{4250}} = \\
 &\quad 0.080 \pm 3(0.0042) \\
 &\quad 0.067 \text{ and } 0.093
 \end{aligned}$$

RESULTS

Lack of control is indicated for all characteristics combined; lot number 12 is outside control limits in a favorable direction and the corresponding results for each of the three components are less than their standard values, Group A being below the lower control limit. For Group A results, lack of control is indicated because lot numbers 10 and 12 are below their lower control limits. Lack of control is indicated for the component characteristics in Group B, because lot numbers 8 and 11 are above their upper control limits. For Group C, lot number 7 is above its upper limit indicating lack of control. Corrective measures are indicated for Groups B and C and steps should be taken to determine whether the Group A component might not be controlled at a smaller value of p_0 , such as 0.06. The values of np_0 for lot numbers 8 and 11 in Group B and lot number 7 in Group C are all larger than 4. The **NOTE** at the end of Section 3.13 does not apply.

Example 20: Control Chart for u , Samples of Unequal Size (Section 3.25)

It is desired to control the number of nonconformities per billet to a standard of 1.000 nonconformity per unit in order that the wire made from such billets of copper will not contain an excessive number of nonconformities. The lot sizes varied greatly from day to day so that a sampling schedule was set up giving three different samples sizes to cover the range of lot sizes received. A control program was instituted using a control chart for nonconformities per unit with reference to the desired standard. Table 43 gives data in terms of nonconformities and nonconformities per unit for 15 consecutive lots under this program. Figure 25 shows the control chart for u .

$$\begin{aligned}
 &\text{Central Line} \\
 &\quad u_0 = 1.000 \\
 &\text{Control Limits} \\
 &\quad n = 100 \\
 &\quad u_0 \pm 3\sqrt{\frac{u_0}{n}} = \\
 &\quad 1.000 \pm 3\sqrt{\frac{1.000}{100}} = \\
 &\quad 1.000 \pm 3(0.100) \\
 &\quad 0.700 \text{ and } 1.300
 \end{aligned}$$

TABLE 43—Lot-by-Lot Inspection Results for Copper Billets in Terms of Number of Nonconformities and Nonconformities per Unit

Lot	Sample Size, n	Number of Nonconformities, c	Nonconformities per Unit, u	Lot	Sample Size, n	Number of Nonconformities, c	Nonconformities per Unit, u
No. 1	100	75	0.750	No. 10	100	130	1.300
No. 2	100	138	1.380	No. 11	100	58	0.580
No. 3	200	212	1.060	No. 12	400	480	1.200
No. 4	400	444	1.110	No. 13	400	316	0.790
No. 5	400	508	1.270	No. 14	200	162	0.810
No. 6	400	312	0.780	No. 15	200	178	0.890
No. 7	200	168	0.840				
No. 8	200	266	1.330	Total	3,500	3,566	
No. 9	100	119	1.190	Overall ^a			1.019

^a $\bar{u} = 3566/3500 = 1.019$.

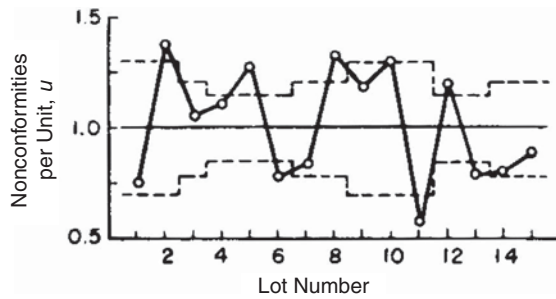


FIG. 25—Control chart for u . Samples of unequal size, $n = 100, 200, 400$; u_0 given.

$$\begin{aligned} n &= 200 \\ u_0 \pm 3\sqrt{\frac{u_0}{n}} &= \\ 1.000 \pm 3\sqrt{\frac{1.000}{200}} &= \\ 1.000 \pm 3(0.0707) &= \\ 0.788 \text{ and } 1.212 \end{aligned}$$

$$\begin{aligned} n &= 400 \\ u_0 \pm 3\sqrt{\frac{u_0}{n}} &= \\ 1.000 \pm 3\sqrt{\frac{1.000}{400}} &= \\ 1.000 \pm 3(0.0500) &= \\ 0.850 \text{ and } 1.150 \end{aligned}$$

RESULTS

Lack of control of quality is indicated with respect to the desired level because lot numbers 2, 5, 8, and 12 are above the upper control limit and lot numbers 6, 11, and 13 are below the lower control limit. The overall level, 1.019 nonconformities per unit, is slightly above the desired value of 1.000 nonconformity per unit. Corrective action is necessary to reduce the spread between successive lots and reduce the average number of nonconformities per unit. The values of np_0 for all lots are at least 100 so that the **NOTE** at end of Section 3.15 does not apply.

Example 21: Control Charts for c , Samples of Equal Size (Section 3.26)

A Type D motor is being produced by a manufacturer that desires to control the number of nonconformities per motor at a level of $u_0 = 3.000$ nonconformities per unit with respect to all visual nonconformities. The manufacturer produces on a continuous basis and decides to take a sample of 25 motors every day, where a day's product is treated as a lot. Because of the nature of the process, plans are to control the product for these nonconformities at a level such that $c_0 = 75.0$ nonconformities and $nu_0 = c_0$. Table 44 gives data in terms of number of nonconformities, c , and also the number of nonconformities per unit, u , for ten consecutive days. Figure 26 shows the control chart for c . As in *Example 20*, a control chart may be made for u , where the central line is $u_0 = 3.000$ and the control limits are

TABLE 44—Daily Inspection Results for Type D Motors in Terms of Nonconformities per Sample and Nonconformities per Unit

Lot	Sample Size, n	Number of Nonconformities, c	Nonconformities per Unit, u
No. 1	25	81	3.24
No. 2	25	64	2.56
No. 3	25	53	2.12
No. 4	25	95	3.80
No. 5	25	50	2.00
No. 6	25	73	2.92
No. 7	25	91	3.64
No. 8	25	86	3.44
No. 9	25	99	3.96
No. 10	25	60	2.40
Total	250	752	30.08
Average	25.0	75.2	3.008

$$\begin{aligned} u_0 \pm 3\sqrt{\frac{u_0}{n}} &= \\ 3.000 \pm 3\sqrt{\frac{3.000}{25}} &= \\ 3.000 \pm 3(0.3464) &= \\ 1.96 \text{ and } 4.04 \end{aligned}$$

$$\begin{aligned} \text{Central Line} \\ c_0 = nu_0 = 3.000 \times 25 = 75.0 \end{aligned}$$

$$\begin{aligned} \text{Control Limits} \\ n &= 25 \\ c_0 \pm 3\sqrt{c_0} &= \\ 75.0 \pm 3\sqrt{75.0} &= \\ 75.0 \pm 3(8.66) &= \\ 49.02 \text{ and } 100.98 \end{aligned}$$

RESULTS

No significant deviations from the desired level. There are no points outside limits so that the **NOTE** at the end of Section 3.16 does not apply. In addition, $c_0 = 75$, larger than 4.

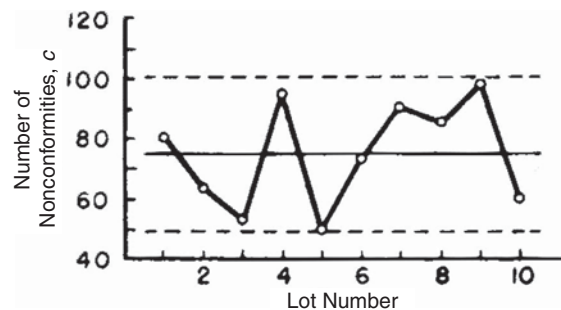


FIG. 26—Control chart for c . Sample of equal size, $n = 25$; c_0 given.

[illegible]

Central Lines

$$\text{For } \bar{X} : \bar{\bar{X}} = 0.7388$$

$$\text{For } R : \bar{R} = 0.186$$

$$\text{For } X : \bar{\bar{X}} = 0.7388$$

Control Limits

$$n = 5$$

$$\text{For } \bar{X} : \bar{\bar{X}} \pm A_2 \bar{R} =$$

$$0.7388 \pm (0.577)(0.186)$$

$$0.631 \text{ and } 0.846$$

$$\text{For } R : D_4 \bar{R} = (2.115)(0.186) = 0.393$$

$$D_3 \bar{R} = (0)(0.186) = 0$$

$$\text{For } X : \pm E_2 \bar{MR} =$$

$$0.7388 \pm (1.290)(0.186)$$

$$0.499 \text{ and } 0.979$$

RESULTS

None of the charts give evidence of lack of control.

Example 23: Control Chart for Individuals, X —Using Rational Subgroups, Standard Given, Based on μ_0 and σ_0 (Section 3.29)

In the hand spraying of small instrument pins held in bar frames of 25 each, coating thickness and weight had to be delicately controlled and spray-gun adjustments were critical

and had to be watched continuously from bar to bar. Weights were measured by careful weighing before and after removal of the coating. Destroying more than one pin per bar was economically not feasible, yet failure to catch a bar departing from standards might result in the unsatisfactory performance of some 24 assembled instruments. The standard lot size for these instrument pins was 100 so that initially control charts for average and range were set up with $n = 4$. It was found that the variation in thickness of coating on the 25 pins on a single bar was quite small as compared with the between-bar variation. Accordingly, as an adjunct to the control charts for average and range, a control chart for individuals, X , at the sprayer position was adopted for the operator's guidance.

Table 46 gives data comprising observations on 32 pins taken from consecutive bar frames together with 8 average and range values where $n = 4$. It was desired to control the weight with an average $\mu_0 = 20.00$ mg and $\sigma_0 = 0.900$ mg. Figure 28 shows the control chart for individual values X for coating weights of instrument pins together with the control charts for \bar{X} and R for samples where $n = 4$.

Central Line

$$\text{For } X : \mu_0 = 20.00$$

Control Limits

$$\text{For } X : \mu_0 \pm 3\sigma_0 =$$

$$20.00 \pm 3(0.900)$$

$$17.3 \text{ and } 22.7$$

TABLE 46—Coating Weights of Instrument Pins, milligrams

		Sample, $n = 4$					Sample, $n = 4$		
Individual	Individual Observation, X	Sample	Average, \bar{X}	Range, R	Individual	Individual Observation, X	Sample	Average, \bar{X}	Range, R
1	18.5	1	18.90	4.7	18	20.6			
2	21.2				19	20.8			
3	19.4				20	21.6			
4	16.5				21	22.8	6	22.80	1.0
5	17.9	2	19.60	3.3	22	22.2			
6	19.0				23	23.2			
7	20.3				24	23.0			
8	21.2				25	19.0	7	19.75	1.5
9	19.6	3	20.08	0.9	26	20.5			
10	19.8				27	20.3			
11	20.4				28	19.2			
12	20.5				29	20.7	8	20.32	1.9
13	22.2	4	21.20	1.9	30	21.0			
14	21.5				31	20.5			
15	20.8				32	19.1			
16	20.3				Total	652.7		163.17	17.7
17	19.1	5	20.52	2.5	Average	20.40		20.40	2.21

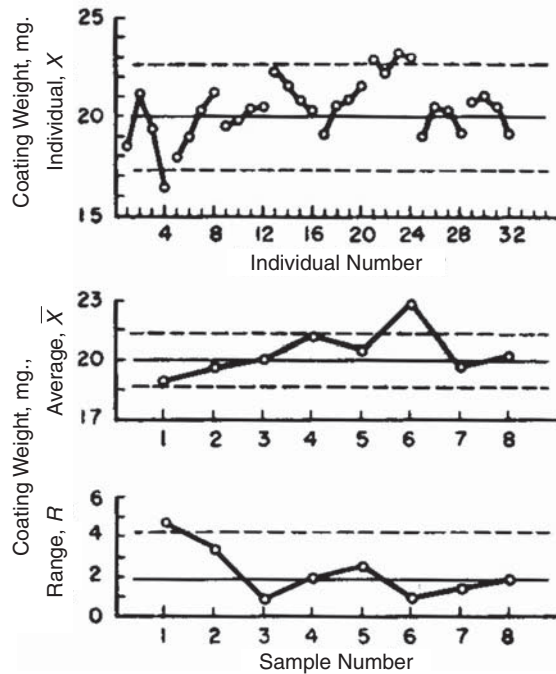


FIG. 28—Control charts for X , \bar{X} , and R . Small samples of equal size, $n = 4$; μ_0 , σ_0 given.

Central Lines

For \bar{X} : $\mu_0 = 20.00$

For R : $d_2\sigma_0 = (2.059)(0.900) = 1.85$

Control Limits

$n = 4/$

For \bar{X} : $\mu_0 \pm A\sigma_0 =$
 $20.00 \pm (1.500)(0.900)$
 $18.65 \text{ and } 21.35$

For R : $D_2\sigma_0 = (4.698)(0.900) = 4.23$

$D_1\sigma_0 = (0)(0.900) = 0$

RESULTS

All three charts show lack of control. At the outset, both the chart for ranges and the chart for individuals gave indications of lack of control. Subsequently, for Sample 6 the control chart for individuals showed the first unit in the sample of 4 to be outside its upper control limit, thus indicating lack of control before the entire sample was obtained.

Example 24: Control Charts for Individuals, X , and Moving Range, MR , of Two Observations, No Standard Given—Based on \bar{X} and MR , the Mean Moving Range (Section 3.30A)

A distilling plant was distilling and blending batch lots of denatured alcohol in a large tank. It was desired to control the percentage of methanol for this process. The variability of sampling within a single lot was found to be negligible so it was decided feasible to take only one observation per lot and to set control limits based on the moving range of successive lots. Table 47 gives a summary of the methanol content, X , of 26 consecutive lots of the denatured alcohol and the 25 values of the moving range, MR , the range of successive lots with $n = 2$. Figure 29 gives control charts for individuals, X , and the moving range, MR .

TABLE 47—Methanol Content of Successive Lots of Denatured Alcohol and Moving Range for $n = 2$

Lot	Percentage of Methanol, X	Moving Range, MR	Lot	Percentage of Methanol, X	Moving Range, MR
No. 1	4.6	...	No. 14	5.5	0.1
No. 2	4.7	0.1	No. 15	5.2	0.3
No. 3	4.3	0.4	No. 16	4.6	0.6
No. 4	4.7	0.4	No. 17	5.5	0.9
No. 5	4.7	0	No. 18	5.6	0.1
No. 6	4.6	0.1	No. 19	5.2	0.4
No. 7	4.8	0.2	No. 20	4.9	0.3
No. 8	4.8	0	No. 21	4.9	0
No. 9	5.2	0.4	No. 22	5.3	0.4
No. 10	5.0	0.2	No. 23	5.0	0.3
No. 11	5.2	0.2	No. 24	4.3	0.7
No. 12	5.0	0.2	No. 25	4.5	0.2
No. 13	5.6	0.6	No. 26	4.4	0.1
			Total	128.1	7.2

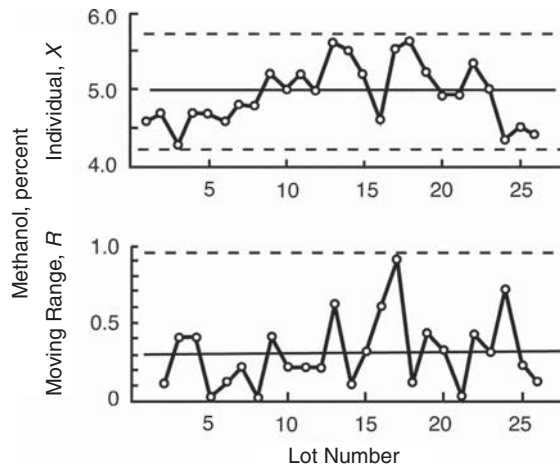


FIG. 29—Control charts for X and MR . No standard given; based on moving range, where $n = 2$.

$$\begin{aligned} \text{Central Lines} \\ \text{For } X: \bar{X} &= \frac{128.1}{26} = 4.927 \\ \text{For } R: \bar{R} &= \frac{7.2}{25} = 0.288 \end{aligned}$$

$$\begin{aligned} \text{Control Limits} \\ n = 2 \\ \text{For } X: \bar{X} \pm E_2 \bar{MR} = \bar{X} \pm 2.660 \bar{MR} = \\ 4.927 \pm (2.660)(0.288) \\ 4.2 \text{ and } 5.7 \end{aligned}$$

$$\begin{aligned} \text{For } R: D_4 \bar{MR} &= (3.267)(0.288) = 0.94 \\ D_3 \bar{MR} &= (0)(0.288) = 0 \end{aligned}$$

RESULTS

The trend pattern of the individuals and their tendency to crowd the control limits suggests that better control may be attainable.

Example 25: Control Charts for Individuals, X , and Moving Range, MR , of Two Observations, Standard Given—Based on μ_0 and σ_0 (Section 3.30B)

The data are from the same source as for Example 24, in which a distilling plant was distilling and blending batch lots of denatured alcohol in a large tank. It was desired to control the percentage of water for this process. The variability of sampling within a single lot was found to be negligible so it was decided to take only one observation per lot and to set control limits for individual values, X , and for the moving range, MR , of successive lots with $n = 2$ where $\mu_0 = 7.800\%$ and $\sigma_0 = 0.200\%$. Table 48 gives a summary of the water content of 26 consecutive lots of the denatured alcohol and the 25 values of the moving range, R . Figure 30 gives control charts for individuals, i , and for the moving range, MR .

$$\begin{aligned} \text{Central Lines} \\ \text{For } X: \mu_0 &= 7.800 \\ n &= 2 \\ \text{For } R: d_2 \sigma_0 &= (1.128)(0.200) = 0.23 \end{aligned}$$

$$\begin{aligned} \text{Control Limits} \\ \text{For } X: \mu_0 \pm 3\sigma &= \\ 7.800 \pm 3(0.200) \\ 7.2 \text{ and } 8.4 \\ n &= 2 \end{aligned}$$

$$\begin{aligned} \text{For } R: D_2 \sigma_0 &= (3.686)(0.200) = 0.74 \\ D_1 \sigma_0 &= (0)(0.200) = 0 \end{aligned}$$

TABLE 48—Water Content of Successive Lots of Denatured Alcohol and Moving Range for $n = 2$

Lot	Percentage of Water, X	Moving Range, MR	Lot	Percentage of Water, X	Moving Range, MR
No. 1	8.9	...	No. 15	8.2	0
No. 2	7.7	1.2	No. 16	7.5	0.7
No. 3	8.2	0.5	No. 17	7.5	0
No. 4	7.9	0.3	No. 18	7.8	0.3
No. 5	8.0	0.1	No. 19	8.5	0.7
No. 6	8.0	0	No. 20	7.5	1.0
No. 7	7.7	0.3	No. 21	8.0	0.5
No. 8	7.8	0.1	No. 22	8.5	0.5
No. 9	7.9	0.1	No. 23	8.4	0.1
No. 10	8.2	0.3	No. 24	7.9	0.5
No. 11	7.5	0.7	No. 25	8.4	0.5
No. 12	7.5	0	No. 26	7.5	0.9
No. 13	7.9	0.4	Total	207.1	10.0
No. 14	8.2	0.3	Number of values	26	25
			Average	7.965	0.400

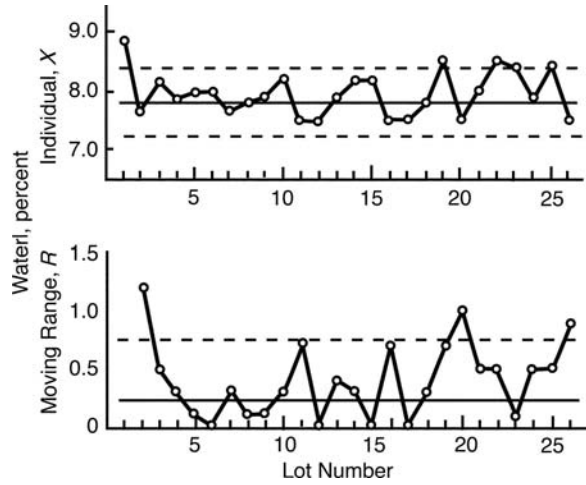


FIG. 30—Control charts for \bar{X} and moving range, MR , where $n = 2$. Standard given; based on μ_0 and σ_0 .

RESULTS

Lack of control at desired levels is indicated with respect to both the individual readings and the moving range. These results indicate corrective measures should be taken to reduce the level in percent and to reduce the variation between lots.

SUPPLEMENT 3.A

Mathematical Relations and Tables of Factors for Computing Control Chart Lines

Scope

Supplement A presents mathematical relations used in arriving at the factors and formulas of **PART 3**. In addition, Supplement A presents approximations to c_4 , $1/c_4$, B_3 , B_4 , B_5 , and B_6 for use when needed. Finally, a more comprehensive tabulation of values of these factors is given in Tables 3.49 and 3.50, including reciprocal values of c_4 and d_2 , and values of d_3 .

Factors c_4 , d_2 , and d_3 , (values for $n = 2$ to 25 , inclusive, in Table 49)

The relations given for factors c_4 , d_2 , and d_3 are based on sampling from a universe having a normal distribution [1, p. 184].

$$c_4 = \sqrt{\frac{2}{n-1} \frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!}} \quad (42)$$

where the symbol $(k/2)!$ is called “ $k/2$ factorial” and satisfies the relations $(-1/2)! = \sqrt{\pi}$, $0! = 1$, and $(k/2)! = (k/2)[(k-2)/2]!$ for $k = 1, 2, 3, \dots$. If k is even, $(k/2)!$ is simply the product of all integers from $k/2$ down to 1; for example, if $k = 8$, $(8/2)! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. If k is odd, $(k/2)!$ is the product of all half-integers from $k/2$ down to $1/2$, multiplied by $\sqrt{\pi}$; for example, if $k = 7$, so $(7/2)! = (7/2) \cdot (5/2) \cdot (3/2) \cdot (1/2) \cdot \sqrt{\pi} = 11.6317$.

$$d_2 = \int_{-\infty}^{\infty} [1 - (1 - \alpha_1)^n - \alpha_1^n] dx_1 \quad (43)$$

where

$$\alpha_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_1} e^{-(x^2/2)} dx, \text{ and } n = \text{sample size.} \quad (44)$$

$$d_3 = \sqrt{2 \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} [1 - \alpha_1^n - (1 - \alpha_n)^n + (\alpha_1 - \alpha_n)^n] dx_n dx_1 - d_2^2}$$

where

$$\alpha_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_1} e^{-(x^2/2)} dx$$

$$\alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_n} e^{-(x^2/2)} dx$$

n = sample size, and d_2 = average range for a normal law distribution with standard deviation equal to unity. (In his original paper, Tippett [10] used w for the range and \bar{w} for d_2 .)

The relations just mentioned for c_4 , d_2 , and d_3 are exact when the original universe is normal but this does not limit their use in practice. They may for most practical purposes be considered satisfactory for use in control chart work although the universe is not Normal. Because the relations are involved and thus difficult to compute, values of c_4 , d_2 , and d_3 for $n = 2$ to 25 , inclusive, are given in Table 49. All values listed in the table were computed to enough significant figures so that when rounded off in accordance with standard practices the last figure shown in the table was not in doubt.

Standard Deviations of \bar{X} , s , R , p , np , u , and c

The standard deviations of \bar{X} , s , R , p , etc., used in setting 3-sigma control limits and designated $\sigma_{\bar{X}}$, σ_s , σ_R , σ_p , etc., in **PART 3**, are the standard deviations of the sampling distributions of \bar{X} , s , R , p , etc., for subgroups (samples) of size n . They are not the standard deviations which might be computed from the subgroup values of \bar{X} , s , R , p , etc., plotted on the control charts but are computed by formula from the quantities listed in Table 51.

The standard deviations $\sigma_{\bar{X}}$ and σ_s computed in this way are unaffected by any assignable causes of variation between subgroups. Consequently, the control charts derived from them will detect assignable causes of this type.

The relations in Eqs 45 to 55, inclusive, which follow, are all of the form standard deviation of the sampling distribution is equal to a function of both the sample size, n , and a universe value σ , p' , u' , or c' .

In practice, a sample estimate or standard value is substituted for σ , p' , u' , or c' . The quantities to be substituted for the cases “no standard given” and “standard given” are shown below immediately after each relation.

Average, \bar{X}

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \quad (45)$$

where σ is the standard deviation of the universe. For no standard given, substitute \bar{s}/c_4 or \bar{R}/d_2 for σ , or for standard given, substitute σ_0 for σ . Equation 45 does not assume a Normal distribution [1, pp. 180–181].

Standard Deviation, s

$$\sigma_s = \sigma \sqrt{1 - c_4^2} \quad (46)$$

or by substituting the expression for c_4 from Equation 42 and noting $((n-1)/2) \times ((n-3)/2)! = ((n-1)/2)!$,

$$\sigma_s = \left(\sqrt{1 - \left[\frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n-1}{2}\right)! \left(\frac{n-3}{2}\right)!} \right]} \right) \sigma \quad (47)$$

TABLE 49—Factors for Computing Control Chart Lines

Observations in Sample, n	Chart for Averages			Chart for Standard Deviations						Chart for Ranges						
	Factors for Control Limits			Factors for Central Line		Factors for Control Limits				Factors for Central Line		Factors for Control Limits				
	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8862	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5908	0.888	0	4.358	0	2.575
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.079	0	2.004
7	1.134	0.419	1.182	0.9594	1.0424	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.205	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.307	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.686	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0253	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0230	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.923	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.740	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637

17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.609
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.733	1.489	5.889	0.404	1.596
20	0.671	0.180	0.680	0.9869	1.0132	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.606	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0120	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.660	5.979	0.435	1.565
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.711	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.032	0.452	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.805	6.056	0.459	1.541
Over 25	$3/\sqrt{n}$...	a	b	c	d	e	f	g

Notes. Values of all factors in this table were recomputed in 1987 by A.T.A. Holden of the Rochester Institute of Technology. The computed values of d_2 and d_3 as tabulated agree with appropriately rounded values from H.L. Harter, in *Order Statistics and Their Use in Testing and Estimation*, Vol. 1, 1969, p. 376.

$$^a 3/\sqrt{n-0.5}$$

$$^b (4n-4)/(4n-3)$$

$$^c (4n-3)/(4n-4)$$

$$^d 1-3/\sqrt{2n-2.5}$$

$$^e 1+3/\sqrt{2n-2.5}$$

$$^f (4n-4)/(4n-3)-3/\sqrt{2n-1.5}$$

$$^g (4n-4)/(4n-3)+3/\sqrt{2n-1.5}$$

See Supplement 3.B, Note 9, on replacing first term in footnotes b , c , f , and g by unity.

TABLE 50—Factors for Computing Control Limits—Chart for Individuals

Observations in Sample, n	Chart for Individuals	
	Factors for Control Limits	
	E_2	E_3
2	2.659	3.760
3	1.772	3.385
4	1.457	3.256
5	1.290	3.192
6	1.184	3.153
7	1.109	3.127
8	1.054	3.109
9	1.010	3.095
10	0.975	3.084
11	0.946	3.076
12	0.921	3.069
13	0.899	3.063
14	0.881	3.058
15	0.864	3.054
16	0.849	3.050
17	0.836	3.047
18	0.824	3.044
19	0.813	3.042
20	0.803	3.040
21	0.794	3.038
22	0.785	3.036
23	0.778	3.034
24	0.770	3.033
25	0.763	3.031
Over 25	$3/d_2$	3

The expression under the square root sign in Eq 47 can be rewritten as the reciprocal of a sum of three terms obtained by applying *Stirling's formula* (see Eq 12.5.3 of [10]) simultaneously to each factorial expression in Eq 47. The result is

$$\sigma_s = \frac{\sigma}{\sqrt{2n - 1.5 + P_n}} \quad (48)$$

where P_n is a relatively small positive quantity which decreases toward zero as n increases. For no standard given, substitute \bar{s}/c_4 or \bar{R}/d_2 for σ ; for standard given, substitute σ_0 for σ . For control chart purposes, these relations may be used for distributions other than normal.

The exact relation of Eq 46 or Eq 47 is used in **PART 3** for control chart analyses involving σ_s and for the determination

TABLE 51—Basis of Standard Deviations for Control Limits

Control Chart	Standard Deviation Used in Computing 3-Sigma Limits Is Computed from	
	Control—No Standard Given	Control—Standard Given
\bar{X}	\bar{s} or \bar{R}	σ_0
s	\bar{s} or \bar{R}	σ_0
R	\bar{s} or \bar{R}	σ_0
p	\bar{p}	p_0
np	$n\bar{p}$	np_0
u	\bar{u}	u_0
c	\bar{c}	c_0

Note. \bar{X} , \bar{R} , etc., are computed averages of subgroup values; σ_0 , p_0 , etc., are standard values.

of factors B_3 and B_4 of Table 6, and of B_5 and B_6 of Table 16.

Range, R

$$\sigma_R = d_3 \sigma \quad (49)$$

where σ is the standard deviation of the universe. For no standard given, substitute \bar{s}/c_4 or \bar{R}/d_2 for σ ; for standard given, substitute σ_0 for σ .

The factor d_3 given in Eq 44 represents the standard deviation for ranges in terms of the true standard deviation of a normal distribution.

Fraction Nonconforming, p

$$\sigma_p = \sqrt{\frac{p'(1-p')}{n}} \quad (50)$$

where p' is the value of the fraction nonconforming for the universe. For no standard given, substitute \bar{p} for p' in Eq 50; for standard given, substitute p_0 for p' . When p' is so small that the factor $(1 - p')$ may be neglected, the following approximation is used

$$\sigma_p = \sqrt{\frac{p'}{n}} \quad (51)$$

Number of Nonconforming Units, np

$$\sigma_{np} = \sqrt{np'(1-p')} \quad (52)$$

where p' is the value of the fraction nonconforming for the universe. For no standard given, substitute \bar{p} for p' ; and for standard given, substitute p for p' . When p' is so small that the term $(1 - p')$ may be neglected, the following approximation is used

$$\sigma_{np} = \sqrt{np'} \quad (53)$$

The quantity np has been widely used to represent the number of nonconforming units for one or more characteristics.

The quantity np has a binomial distribution. Equations 50 and 52 are based on the binomial distribution in which the theoretical frequencies for $np = 0, 1, 2, \dots, n$ are given by the first, second, third, etc. terms of the expansion of the binomial $[(1 - p')n]$ where p' is the universe value.

Nonconformities per Unit, u

$$\sigma_u = \sqrt{\frac{u'}{n}} \quad (54)$$

where n is the number of units in sample, and u' is the value of *nonconformities per unit* for the universe. For no standard given, substitute \bar{u} for u' ; for standard given, substitute u_0 for u' .

The number of nonconformities found on any one unit may be considered to result from an unknown but large (practically infinite) number of causes where a nonconformity could possibly occur combined with an unknown but very small probability of occurrence due to any one point. This leads to the use of the Poisson distribution for which the standard deviation is the square root of the expected number of nonconformities on a single unit. This distribution is likewise applicable to sums of such numbers, such as the observed values of c , and to averages of such numbers, such as observed values of u , the standard deviation of the averages being $1/n$ times that of the sums. Where the number of nonconformities found on any one unit results from a known number of potential causes (relatively a small number as compared with the case described above), and the distribution of the nonconformities per unit is more exactly a multinomial distribution, the Poisson distribution, although an approximation, may be used for control chart work in most instances.

Number of Nonconformities, c

$$\sigma_c = \sqrt{nu'} = \sqrt{c'} \quad (55)$$

where n is the number of units in sample, u' is the value of *nonconformities per unit* for the universe, and c' is the number of nonconformities in samples of size n for the universe. For no standard given, substitute $\bar{c} = n\bar{u}$ for c' ; for standard given, substitute $c'_0 = nu'_0$ for c' . The distribution of the observed values of c is discussed above.

FACTORS FOR COMPUTING CONTROL LIMITS

Note that all these factors are actually functions of n only, the constant 3 resulting from the choice of 3-sigma limits.

Averages

$$A = \frac{3}{\sqrt{n}} \quad (56)$$

$$A_3 = \frac{3}{c_4\sqrt{n}} \quad (57)$$

$$A_2 = \frac{3}{d_2\sqrt{n}} \quad (58)$$

NOTE— $A_3 = A/c_4$, $A_2 = A/d_2$.

Standard deviations

$$B_5 = c_4 - 3\sqrt{1 - c_4^2} \quad (59)$$

$$B_6 = c_4 + 3\sqrt{1 - c_4^2} \quad (60)$$

$$B_3 = 1 - \frac{3}{c_4}\sqrt{1 - c_4^2} \quad (61)$$

$$B_4 = 1 + \frac{3}{c_4}\sqrt{1 - c_4^2} \quad (62)$$

NOTE— $B_3 = B_5/c_4$, $B_4 = B_6/c_4$.

Ranges

$$D_1 = d_2 - 3d_3 \quad (63)$$

$$D_2 = d_2 - 3d_3 \quad (64)$$

$$D_3 = 1 - 3\frac{d_3}{d_2} \quad (65)$$

$$D_4 = 1 + 3\frac{d_3}{d_2} \quad (66)$$

NOTE— $D_3 = D_1/d_2$, $D_4 = D_2/d_2$

Individuals

$$E_3 = \frac{3}{c_4} \quad (67)$$

$$E_2 = \frac{3}{d_2} \quad (68)$$

APPROXIMATIONS TO CONTROL CHART FACTORS FOR STANDARD DEVIATIONS

At times it may be appropriate to use approximations to one or more of the control chart factors c_4 , $1/c_4$, B_3 , B_4 , B_5 , and B_6 (see Supplement B, Note 8).

The theory leading to Eqs 47 and 48 also leads to the relation

$$c_4 = \sqrt{\frac{2n - 2.5}{2n - 1.5}} [1 + (0.046875 + Q_n)/n^3] \quad (69)$$

where Q_n is a small positive quantity which decreases towards zero as n increases. Equation 69 leads to the approximation

$$c_4 \doteq \sqrt{\frac{2n - 2.5}{2n - 1.5}} = \sqrt{\frac{4n - 5}{4n - 3}} \quad (70)$$

which is accurate to 3 decimal places for n of 7 or more, and to 4 decimal places for n of 13 or more. The corresponding approximation for $1/c_4$ is

$$1/c_4 \doteq \sqrt{\frac{2n - 1.5}{2n - 2.5}} = \sqrt{\frac{4n - 3}{4n - 5}} \quad (71)$$

which is accurate to 3 decimal places for n of 8 or more, and to 4 decimal places for n of 14 or more. In many

applications, it is sufficient to use the slightly simpler and slightly less accurate approximation

$$c_4 \doteq (4n - 4)/(4n - 3), \quad (72)$$

which is accurate to within one unit in the third decimal place for n of 5 or more, and to within one unit in the fourth decimal place for n of 16 or more [2, p. 34]. The corresponding approximation to $1/c_4$ is

$$1/c_4 \doteq (4n - 3)/(4n - 4), \quad (73)$$

which has accuracy comparable to that of Eq 72.

Note

The approximations to c_4 in Eqs 70 and 72 have the exact relation where

$$\frac{\sqrt{4n-5}}{\sqrt{4n-3}} = \frac{4n-4}{4n-3} \cdot \sqrt{1 - \frac{1}{(4n-4)^2}}$$

The square root factor is greater than 0.998 for n of 5 or more. For n of 4 or more, an even closer approximation to c_4 than those of Eqs 70 and 72 is $(4n - 45)/(4n - 35)$. While the increase in accuracy over Eq 70 is immaterial, this approximation does not require a square root operation.

From Eqs 70 and 3.71

$$\sqrt{1 - c_4^2} \doteq 1/\sqrt{2n - 1.5} \quad (74)$$

and

$$\frac{1}{c_4} \sqrt{1 - c_4^2} \doteq 1/\sqrt{2n - 2.5} \quad (75)$$

If the approximations of Eqs 72, 74, and 75 are substituted into Eqs 59, 60, 61, and 62, the following approximations to the B -factors are obtained:

$$B_5 \cong \frac{4n-4}{4n-3} - \frac{3}{\sqrt{2n-1.5}} \quad (76)$$

$$B_6 \cong \frac{4n-4}{4n-3} + \frac{3}{\sqrt{2n-1.5}} \quad (77)$$

$$B_3 \cong 1 - \frac{3}{\sqrt{2n-1.5}} \quad (78)$$

$$B_4 \cong 1 + \frac{3}{\sqrt{2n-1.5}} \quad (79)$$

With a few exceptions the approximations in Eqs 76, 77, 78, and 79 are accurate to 3 decimal places for n of 13 or more. The exceptions are all one unit off in the third decimal place. That degree of inaccuracy does not limit the practical usefulness of these approximations when n is 25 or more. (See Supplement B, Note 8.) For other approximations to B_5 and B_6 , see Supplement B, Note 9.

Tables 6, 16, 49, and 50 of **PART 3** give all control chart factors through $n = 25$. The factors c_4 , $1/c_4$, B_5 , B_6 , B_3 , and B_4 may be calculated for larger values of n accurately to the same number of decimal digits as the tabled values by using Eqs 70, 71, 76, 77, 78, and 79, respectively. If three-digit accuracy suffices for c_4 or $1/c_4$, Eq 72 or 73 may be used for values of n larger than 25.

SUPPLEMENT 3.B

Explanatory Notes

Note 1

As explained in detail in Supplement 3.A, $\sigma_{\bar{x}}$ and σ_s are based (1) on variation of individual values *within* subgroups and the size n of a subgroup for the first use (A) Control—No Standard Given, and (2) on the adopted standard value of σ and the size n of a subgroup for the second use (B) Control with Respect to a Given Standard. Likewise, for the first use, σ_p is based on the average value of p , designated \bar{p} , and n , and for the second use from p_0 and n . The method for determining σ_R is outlined in Supplement 3.A. For purpose (A), the σ_s must be estimated from the data.

Note 2

This is discussed fully by Shewhart [1]. In some situations in industry in which it is important to catch trouble even if it entails a considerable amount of otherwise unnecessary investigation, 2-sigma limits have been found useful. The necessary changes in the factors for control chart limits will be apparent from their derivation in the text and in Supplement 3.A. Alternatively, in process quality control work, probability control limits based on percentage points are sometimes used [2, pp. 15-16].

Note 3

From the viewpoint of the theory of estimation, if normality is assumed, an unbiased and efficient estimate of the standard deviation within subgroups is

$$\frac{1}{c_4} \sqrt{\frac{(n_1 - 1)s_1^2 + \cdots + (n_k - 1)s_k^2}{n_1 + \cdots + n_k - k}} \quad (80)$$

where c_4 is to be found from Table 6, corresponding to $n = n_1 + \cdots + n_k - k + 1$. Actually, c_4 will lie between .99 and unity if $n_1 + \cdots + n_k - k + 1$ is as large as 26 or more as it usually is, whether n_1 , n_2 , etc. be large, small, equal, or unequal.

Equations 4, 6, and 9, and the procedure of Sections 8 and 9, "Control—No Standard Given," have been adopted for use in **PART 3** with practical considerations in mind, Eq 6 representing a departure from that previously given. From the viewpoint of the theory of estimation they are unbiased or nearly so when used with the appropriate factors as described in the text and for normal distributions are nearly as efficient as Eq 80.

It should be pointed out that the problem of choosing a control chart criterion for use in "Control—No Standard Given" is not essentially a problem in estimation. The criterion is by nature more a test of consistency of the data themselves and must be based on the data at hand including some which may have been influenced by the assignable causes which it is desired to detect. The final justification of a control chart criterion is its proven ability to detect assignable causes economically under practical conditions.

When control has been achieved and standard values are to be based on the observed data, the problem is more a problem in estimation, although in practice many of the assumptions made in estimation theory are imperfectly met and practical considerations, sampling trials, and experience are deciding factors.

In both cases, data are usually plentiful and efficiency of estimation a minor consideration.

Note 4

If most of the samples are of approximately equal size⁴, effort may be saved by first computing and plotting approximate control limits based on some typical sample size, such as the most frequent sample size, standard sample size, or the average sample size. Then, for any point questionably near the limits, the correct limits based on the actual sample size for the point should be computed and also plotted, if the point would otherwise be shown in incorrect relation to the limits.

Note 5

Here it is of interest to note the nature of the statistical distributions involved, as follows.

- (a) With respect to a characteristic for which it is possible for only one nonconformity to occur on a unit, and, in general, when the result of examining a unit is to classify it as nonconforming or conforming by any criterion, the underlying distribution function may often usefully be assumed to be the binomial, where p is the fraction nonconforming and n is the number of units in the sample (for example, see Eq 14 in **PART 3**).
- (b) With respect to a characteristic for which it is possible for two, three, or some other limited number of defects to occur on a unit, such as poor soldered connections on a unit of wired equipment, where we are primarily concerned with the classification of soldered connections, rather than units, into nonconforming and conforming, the underlying distribution may often usefully be assumed to be the binomial, where p is the ratio of the observed to the possible number of occurrences of defects in the sample and n is the possible number of occurrences of defects in the sample instead of the sample size (for example, see Eq 14 in this part, with n defined as number of possible occurrences per sample).
- (c) With respect to a characteristic for which it is possible for a large but indeterminate number of nonconformities to occur on a unit, such as finish defects on a painted surface, the underlying distribution may often usefully be assumed to be the Poisson distribution. (The proportion of nonconformities expected in the sample, p , is indeterminate and usually small; and the possible number of occurrences of nonconformities in the sample, n , is also indeterminate and usually large; but the product np is finite. For the sample this np value is c .) (For example, see Eq 22 in **PART 3**.) For characteristics of types (a) and (b), the fraction p is almost invariably small, say less than 0.10, and under these circumstances the Poisson distribution may be used as a satisfactory approximation to the binomial. Hence, in general, for all these three types of characteristics, taken individually or collectively, we may use relations based on the Poisson distribution. The relations given for control limits for number of nonconformities (Sections 3.16 and 3.26) have accordingly been based

directly on the Poisson distribution, and the relations for control limits for nonconformities per unit (Sections 3.15 and 3.25), have been based indirectly thereon.

Note 6

In the control of a process, it is common practice to extend the central line and control limits on a control chart to cover a future period of operations. This practice constitutes control with respect to a standard set by previous operating experience and is a simple way to apply this principle when no change in sample size or sizes is contemplated.

When it is not convenient to specify the sample size or sizes in advance, standard values of μ , σ , etc. may be derived from past control chart data using the relations

$$\begin{aligned}\mu_0 &= \bar{\bar{X}} = \bar{X} \text{ (if individual chart)} & np_0 &= n\bar{p} \\ \sigma_0 &= \frac{\bar{R}}{d_2} \text{ or } \frac{\bar{s}}{c_4} = \frac{\overline{MR}}{d_2} \text{ (if ind. chart)} & u_0 &= \bar{u} \\ vp_0 &= \bar{p} & c_0 &= \bar{c}\end{aligned}$$

where the values on the right-hand side of the relations are derived from past data. In this process a certain amount of arbitrary judgment may be used in omitting data from subgroups found or believed to be out of control.

Note 7

It may be of interest to note that, for a given set of data, the mean moving range as defined here is the average of the two values of \bar{R} which would be obtained using ordinary ranges of subgroups of two, starting in one case with the first observation and in the other with the second observation.

The mean moving range is capable of much wider definition [12], but that given here has been the one used most in process quality control.

When a control chart for averages and a control chart for ranges are used together, the chart for ranges gives information which is not contained in the chart for averages and the combination is very effective in process control. The combination of a control chart for individuals and a control chart for moving ranges does not possess this dual property; all the information in the chart for moving ranges is contained, somewhat less explicitly, in the chart for individuals.

Note 8

The tabled values of control chart factors in this Manual were computed as accurately as needed to avoid contributing materially to rounding error in calculating control limits. But these limits also depend: (1) on the factor 3—or perhaps 2—based on an empirical and economic judgment, and (2) on data that may be appreciably affected by measurement error. In addition, the assumed theory on which these factors are based cannot be applied with unerring precision. Somewhat cruder approximations to the exact theoretical values are quite useful in many practical situations. The form of approximation, however, must be simple to use and

⁴ According to Ref. 11, p. 18, "If the samples to be used for a p -chart are not of the same size, then it is sometimes permissible to use the *average sample size* for the series in calculating the control limits." As a rule of thumb, the authors propose that this approach works well as long as, "the largest sample size is no larger than twice the average sample size, and the smallest sample size is no less than half the average sample size." Any samples, whose sample sizes are outside this range, should either be separated (if too big) or combined (if too small) in order to make them of comparable size. Otherwise, the only other option is to compute control limits based on the actual sample size for each of these affected samples.

reasonably consistent with the theory. The approximations in **PART 3**, including Supplement 3.A, were chosen to satisfy these criteria with little loss of numerical accuracy.

Approximate formulas for the values of control chart factors are most often useful under one or both of the following conditions: (1) when the subgroup sample size n exceeds the largest sample size for which the factor is tabled in this Manual; or (2) when exact calculation by computer program or by calculator is considered too difficult.

Under one or both of these conditions the usefulness of approximate formulas may be affected by one or more of the following: (a) there is unlikely to be an economically justifiable reason to compute control chart factors to *more* decimal places than given in the tables of this Manual; it may be equally satisfactory in most practical cases to use an approximation having a decimal-place accuracy not much less than that of the tables, for instance, one having a known maximum error in the same final decimal place; (b) the use of factors involving the sample range in samples larger than 25 is inadvisable; (c) a computer (with appropriate software) or even some models of pocket calculator may be able to compute from an exact formula by subroutines so fast that little or nothing is gained either by approximating the exact formula or by storing a table in memory; (d) because some approximations suitable for large sample sizes are unsuitable for small ones, computer programs using approximations for control chart factors may require conditional branching based on sample size.

Note 9

The value of c_4 rises towards unity as n increases. It is then reasonable to replace c_4 by unity if control limit calculations can thereby be significantly simplified with little loss of numerical accuracy. For instance, Eqs 4 and 6 for samples of 25 or more ignore c_4 factors in the calculation of \bar{s} . The maximum absolute percentage error in width of the control limits on \bar{X} or s is not more than 100 $(1 - c_4)$ %, where c_4 applies to the smallest sample size used to calculate \bar{s} .

Previous versions of this Manual gave approximations to B_5 and B_6 , which substituted unity for c_4 and used $2(n - 1)$ instead of $2n - 1.5$ in the expression under the square root sign of Eq 74. These approximations were judged appropriate compromises between accuracy and simplicity. In recent years three changes have occurred: (a) simple, accurate and inexpensive calculators have become widely available; (b) closer but still quite simple approximations to B_5 and B_6 have been devised; and (c) some applications of assigned standards stress the desirability of having numerically accurate limits. (See *Examples 12 and 13*.)

There thus appears to be no longer any practical simplification to be gained from using the previously published approximations for B_5 and B_6 . The substitution of unity for c_4 shifts the value for the central line upward by approximately $(25/n)$ %; the substitution of $2(n - 1)$ for $2n - 1.5$ increases the width between control limits by approximately $(12/n)$ %. Whether either substitution is material depends on the application.

References

- [1] Shewhart, W.A., *Economic Control of Quality of Manufactured Product*, Van Nostrand, New York, 1931; republished by ASQC Quality Press, Milwaukee, WI, 1980.
- [2] American National Standards Z1.1-1985 (ASQC B1-1985), "Guide for Quality Control Charts," Z1.2-1985 (ASQC B2-1985), "Control Chart Method of Analyzing Data," Z1.3-1985 (ASQC B3-1985), "Control Chart Method of Controlling Quality During Production," American Society for Quality Control, Nov. 1985, Milwaukee, WI, 1985.
- [3] Simon, L.E., *An Engineer's Manual of Statistical Methods*, Wiley, New York, 1941.
- [4] British Standard 600:1935, Pearson, E.S., "The Application of Statistical Methods to Industrial Standardization and Quality Control," British Standard 600 R:1942, Dudding, B.P. and Jennett, W.J., "Quality Control Charts," British Standards Institution, London, England.
- [5] Bowker, A.H. and Lieberman, G.L., *Engineering Statistics*, 2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1972.
- [6] Burr, I.W., *Engineering Statistics and Quality Control*, McGraw-Hill, New York, 1953.
- [7] Duncan, A.J., *Quality Control and Industrial Statistics*, 5th ed., Irwin, Homewood, IL, 1986.
- [8] Grant, E.L. and Leavenworth, R.S., *Statistical Quality Control*, 5th ed., McGraw-Hill, New York, 1980.
- [9] Ott, E.R., Schilling, E.G., and Neubauer, D.V., *Process Quality Control*, 4th ed., McGraw-Hill, New York, 2005.
- [10] Tippett, L.H.C., "On the Extreme Individuals and the Range of Samples Taken from a Normal Population," *Biometrika*, Vol. 17, 1925, pp. 364-387.
- [11] Small, B.B., ed., *Statistical Quality Control Handbook*, AT&T Technologies, Indianapolis, IN, 1984.
- [12] Hoel, P.G., "The Efficiency of the Mean Moving Range," *Ann. Math. Stat.*, Vol. 17, No. 4, Dec. 1946, pp. 475-482.

Selected Papers on Control Chart Techniques⁵

A. General

- Alwan, L.C. and Roberts, H.V., "Time-Series Modeling for Statistical Process Control," *J. Bus. Econ. Stat.*, Vol. 6, 1988, pp. 393-400.
- Barnard, G.A., "Control Charts and Stochastic Processes," *J. R. Stat. Soc. Ser. B*, Vol. 21, 1959, pp. 239-271.
- Ewan, W.D. and Kemp, K.W., "Sampling Inspection of Continuous Processes with No Autocorrelation Between Successive Results," *Biometrika*, Vol. 47, 1960, p. 363.
- Freund, R.A., "A Reconsideration of the Variables Control Chart," *Indust. Qual. Control*, Vol. 16, No. 11, May 1960, pp. 35-41.
- Gibra, I.N., "Recent Developments in Control Chart Techniques," *J. Qual. Technol.*, Vol. 7, 1975, pp. 183-192.
- Vance, L.C., "A Bibliography of Statistical Quality Control Chart Techniques, 1970-1980," *J. Qual. Technol.*, Vol. 15, 1983, pp. 59-62.

B. Cumulative Sum (CUSUM) Charts

- Crosier, R.B., "A New Two-Sided Cumulative Sum Quality-Control Scheme," *Technometrics*, Vol. 28, 1986, pp. 187-194.
- Crosier, R.B., "Multivariate Generalizations of Cumulative Sum Quality-Control Schemes," *Technometrics*, Vol. 30, 1988, pp. 291-303.
- Goel, A.L. and Wu, S.M., "Determination of A. R. L. and A Contour Nomogram for CUSUM Charts to Control Normal Mean," *Technometrics*, Vol. 13, 1971, pp. 221-230.
- Johnson, N.L. and Leone, F.C., "Cumulative Sum Control Charts—Mathematical Principles Applied to Their Construction and Use," *Indust. Qual. Control*, June 1962, pp. 15-21; July 1962; pp. 29-36; and Aug. 1962, pp. 22-28.
- Johnson, R.A. and Bagshaw, M., "The Effect of Serial Correlation on the Performance of CUSUM Tests," *Technometrics*, Vol. 16, 1974, pp. 103-112.

⁵ Used more for control purposes than data presentation. This selection of papers illustrates the variety and intensity of interest in control chart methods. They differ widely in practical value.

- Kemp, K.W., "The Average Run Length of the Cumulative Sum Chart When a V-Mask is Used," *J. R. Stat. Soc. Ser. B*, Vol. 23, 1961, pp. 149–153.
- Kemp, K.W., "The Use of Cumulative Sums for Sampling Inspection Schemes," *Appl. Stat.*, Vol. 11, 1962, pp. 16–31.
- Kemp, K.W., "An Example of Errors Incurred by Erroneously Assuming Normality for CUSUM Schemes," *Technometrics*, Vol. 9, 1967, pp. 457–464.
- Kemp, K.W., "Formal Expressions Which Can Be Applied in CUSUM Charts," *J. R. Stat. Soc. Ser. B*, Vol. 33, 1971, pp. 331–360.
- Lucas, J.M., "The Design and Use of V-Mask Control Schemes," *J. Qual. Technol.*, Vol. 8, 1976, pp. 1–12.
- Lucas, J.M. and Crosier, R.B., "Fast Initial Response (FIR) for Cumulative Sum Quantity Control Schemes," *Technometrics*, Vol. 24, 1982, pp. 199–205.
- Page, E.S., "Cumulative Sum Charts," *Technometrics*, Vol. 3, 1961, pp. 1–9.
- Vance, L., "Average Run Lengths of Cumulative Sum Control Charts for Controlling Normal Means," *J. Qual. Technol.*, Vol. 18, 1986, pp. 189–193.
- Woodall, W.H. and Ncube, M.M., "Multivariate CUSUM Quality-Control Procedures," *Technometrics*, Vol. 27, 1985, pp. 285–292.
- Woodall, W.H., "The Design of CUSUM Quality Charts," *J. Qual. Technol.*, Vol. 18, 1986, pp. 99–102.

C. Exponentially Weighted Moving Average (EWMA) Charts

- Cox, D.R., "Prediction by Exponentially Weighted Moving Averages and Related Methods," *J. R. Stat. Soc. Ser. B*, Vol. 23, 1961, pp. 414–422.
- Crowder, S.V., "A Simple Method for Studying Run-Length Distributions of Exponentially Weighted Moving Average Charts," *Technometrics*, Vol. 29, 1987, pp. 401–408.
- Hunter, J.S., "The Exponentially Weighted Moving Average," *J. Qual. Technol.*, Vol. 18, 1986, pp. 203–210.
- Roberts, S.W., "Control Chart Tests Based on Geometric Moving Averages," *Technometrics*, Vol. 1, 1959, pp. 239–250.

D. Charts Using Various Methods

- Beneke, M., Leemis, L.M., Schlegel, R.E., and Foote, F.L., "Spectral Analysis in Quality Control: A Control Chart Based on the Periodogram," *Technometrics*, Vol. 30, 1988, pp. 63–70.
- Champ, C.W. and Woodall, W.H., "Exact Results for Shewhart Control Charts with Supplementary Runs Rules," *Technometrics*, Vol. 29, 1987, pp. 393–400.
- Ferrell, E.B., "Control Charts Using Midranges and Medians," *Indust. Qual. Control*, Vol. 9, 1953, pp. 30–34.

- Ferrell, E.B., "Control Charts for Log-Normal Universes," *Indust. Qual. Control*, Vol. 15, 1958, pp. 4–6.
- Hoadley, B., "An Empirical Bayes Approach to Quality Assurance," *ASQC 33rd Annual Technical Conference Transactions*, May 14–16, 1979, pp. 257–263.
- Jaehn, A.H., "Improving QC Efficiency with Zone Control Charts," *ASQC Quality Congress Transactions*, Minneapolis, MN, 1987.
- Langenberg, P. and Iglewicz, B., "Trimmed \bar{X} and R Charts," *Journal of Quality Technology*, Vol. 18, 1986, pp. 151–161.
- Page, E.S., "Control Charts with Warning Lines," *Biometrika*, Vol. 42, 1955, pp. 243–254.
- Reynolds, M.R., Jr., Amin, R.W., Arnold, J.C., and Nachlas, J.A., " \bar{X} Charts with Variable Sampling Intervals," *Technometrics*, Vol. 30, 1988, pp. 181–192.
- Roberts, S.W., "Properties of Control Chart Zone Tests," *Bell System Technical J.*, Vol. 37, 1958, pp. 83–114.
- Roberts, S.W., "A Comparison of Some Control Chart Procedures," *Technometrics*, Vol. 8, 1966, pp. 411–430.

E. Special Applications of Control Charts

- Case, K.E., "The p Control Chart Under Inspection Error," *J. Qual. Technol.*, Vol. 12, 1980, pp. 1–12.
- Freund, R.A., "Acceptance Control Charts," *Indust. Qual. Control*, Vol. 14, No. 4, Oct. 1957, pp. 13–23.
- Freund, R.A., "Graphical Process Control," *Indust. Qual. Control*, Vol. 18, No. 7, Jan. 1962, pp. 15–22.
- Nelson, L.S., "An Early-Warning Test for Use with the Shewhart p Control Chart," *J. Qual. Technol.*, Vol. 15, 1983, pp. 68–71.
- Nelson, L.S., "The Shewhart Control Chart-Tests for Special Causes," *J. Qual. Technol.*, Vol. 16, 1984, pp. 237–239.

F. Economic Design of Control Charts

- Banerjee, P.K. and Rahim, M.A., "Economic Design of \bar{X} -Control Charts Under Weibull Shock Models," *Technometrics*, Vol. 30, 1988, pp. 407–414.
- Duncan, A.J., "Economic Design of \bar{X} Charts Used to Maintain Current Control of a Process," *J. Am. Stat. Assoc.*, Vol. 51, 1956, pp. 228–242.
- Lorenzen, T.J. and Vance, L.C., "The Economic Design of Control Charts: A Unified Approach," *Technometrics*, Vol. 28, 1986, pp. 3–10.
- Montgomery, D.C., "The Economic Design of Control Charts: A Review and Literature Survey," *J. Qual. Technol.*, Vol. 12, 1989, pp. 75–87.
- Woodall, W.H., "Weakness of the Economic Design of Control Charts," (Letter to the Editor, with response by T. J. Lorenzen and L. C. Vance), *Technometrics*, Vol. 28, 1986, pp. 408–410.

4

Measurements and Other Topics of Interest

GLOSSARY OF TERMS AND SYMBOLS USED IN PART 4

In general, the terms and symbols used in **PART 4** have the same meanings as in preceding parts of the Manual. In a few cases, which are indicated in the following glossary, a more specific meaning is attached to them for the convenience of a portion or all of **PART 4**.

GLOSSARY OF TERMS

appraiser, n.—individual person who uses a measurement system. Sometimes the term “operator” is used.

appraiser variation (AV), n.—variation in measurement resulting when different operators use the same measurement system.

capability indices, n.—indices C_p and C_{pk} , which represent measures of process capability compared to one or more specification limits.

equipment variation (EV), n.—variation among measurements of the same object by the same appraiser under the same conditions using the same device.

gage, n.—device used for the purpose of obtaining a measurement.

gage bias, n.—absolute difference between the average of a group of measurements of the same part measured under the same conditions and the true or reference value for the object measured.

gage stability, n.—refers to constancy of bias with time.

gage consistency, n.—refers to constancy of repeatability error with time.

gage linearity, n.—change in bias over the operational range of the gage or measurement system used.

gage repeatability, n.—component of variation due to random measurement equipment effects (EV).

gage reproducibility, n.—component of variation due to the operator effect (AV).

gage R&R, n.—combined effect of repeatability and reproducibility.

gage resolution, n.—refers to the system’s discriminating ability to distinguish between different objects.

long-term variability, n.—accumulated variation from individual measurement data collected over an extended period of time. If measurement data are represented as $x_1, x_2, x_3, \dots, x_n$, the long-term estimate of variability is the ordinary sample standard deviation, s , computed from n individual measurements. For a long enough time period, this standard deviation contains the several long-term effects on variability such as a) material lot-to-lot changes, operator changes, shift-to-shift differences, tool or equipment wear, process drift, environmental changes, measurement and calibration effects among others. The symbol used to stand for this measure is σ_{lt} .

measurement, n.—number assigned to an object representing some physical characteristic of the object for

example density, melting temperature, hardness, diameter, and tensile strength.

measurement system, n.—collection of factors that contribute to a final measurement including hardware, software, operators, environmental factors, methods, time, and objects that are measured. Sometimes the term “measurement process” is used.

performance indices, n.—indices P_p and P_{pk} , which represent measures of process performance compared to one or more specification limits.

process capability, n.—total spread of a stable process using the natural or inherent process variation. The measure of this natural spread is taken as $6\sigma_{st}$, where σ_{st} is the estimated short-term estimate of the process standard deviation.

process performance, n.—total spread of a stable process using the long-term estimate of process variation. The measure of this spread is taken as $6\sigma_{lt}$, where σ_{lt} is the estimated long-term process standard deviation.

short-term variability, n.—estimate of variability over a short interval of time (minutes, hours, or a few batches). Within this time period, long-term effects such as material lot changes, operator changes, shift-to-shift differences, tool or equipment wear, process drift, and environmental changes, among others, are NOT at play. The standard deviation for short-term variability may be calculated from the within subgroup variability estimate when a control chart technique is used. This short-term estimate of variation is dependent of the manner in which the subgroups were constructed. The symbol used to stand for this measure is σ_{st} .

statistical control, n.—process is said to be in a state of statistical control if variation in the process output exhibits a stable pattern and is predictable within limits. In this sense, stability, statistical control, and predictability all mean the same thing when describing the state of a process. Generally, the state of statistical control is established using a control chart technique.

GLOSSARY OF SYMBOLS

Symbol	In PART 4, Measurements
u	smallest degree of resolution in a measurement system
σ	standard deviation of gage repeatability
σ_{st}	short-term standard deviation of a process
σ_{lt}	long-term standard deviation of a process

Symbol	In PART 4, Measurements
θ	standard deviation of reproducibility
τ	standard deviation of the true objects measured
v	standard deviation of measurements, y
y	measurement
x	true value of an object
\bar{x}	process average (location)
e	observed repeatability error term
ε	theoretical random repeatability term in a measurement model
\bar{R}	average range of subgroup data from a control chart
\overline{MR}	average moving range of individual data from a control chart
$q1, q2, q3$	used to stand for various formulations of sums of squares in MSA analysis
α	theoretical random reproducibility term in a measurements model
B	bias
C_p	process capability index
C_{pk}	process capability index adjusted for location (process average)
D	discrimination ratio
PC	process capability ratio
P_p	process performance index
P_{pk}	process performance index adjusted for location (process average)

THE MEASUREMENT SYSTEM

4.1. INTRODUCTION

A measurement system may be described as the total of hardware, software, methods, appraisers (analysts or operators), environmental conditions, and the objects measured that come together to produce a measurement. We can conceive of the combination of all of these factors with time as a measurement process. A measurement process, then, is just a process whose end product is a supply of numbers called measurements. The terms “measurement system” and “measurement process” are used interchangeably.

For any given measurement or set of measurements, we can consider the quality of the measurements themselves and the quality of the process that produced the measurements. The study of measurement quality characteristics and the associate measurement process is referred to as measurement systems analysis (MSA). This field is quite extensive and encompasses a huge range of topics. In this section, we give an overview of several important concepts related to measurement quality. The term “object” is here used to denote that which is measured.

4.2. BASIC PROPERTIES OF A MEASUREMENT PROCESS

There are several basic properties of measurement systems that are widely recognized among practitioners: repeatability, reproducibility, linearity, bias, stability, consistency, and resolution. In studying one or more of these properties, the final result of any such study is some assessment of the capability of the measurement system with respect to the property under investigation. Capability may be cast in several ways, and this may also be application dependent. One of the primary objectives in any MSA effort is to assess variation attributable to the various factors of the system. All of the basic properties assess variation in some form.

Repeatability is the variation that results when a single object is repeatedly measured in the same way, by the same appraiser, under the same conditions using the same measurement system. The term “precision” may also denote this same concept in some quarters, but “repeatability” is found more often in measurement applications. The term “conditions” is sometimes attached to repeatability to denote “repeatability conditions” (see ASTM E456, Standard Terminology Relating to Quality and Statistics). The phrase “Intermediate Precision” is also used (see, for example, ASTM E177, Standard Practice for Use of the Terms Precision and Bias in ASTM Test Methods). The user of a measurement system must decide what constitutes repeatability conditions or intermediate precision for the given application. In assessing repeatability we seek an estimate of the standard deviation, σ , of this type of random error.

Bias is the difference between an accepted reference or standard value for an object and the average value of a sample of several of the object’s measurements under a fixed set of conditions. Sometimes the term “true value” is used in place of reference value. The terms “reference value” or “true value” may be thought of as the most accurate value that can be assigned to the object (often a value made by the best measurement system available for the purpose). Figure 1 illustrates the repeatability and bias concepts.

A closely related concept is *linearity*. This is defined as a change in measurement system bias as the object’s true or reference value changes. “Smaller” objects may exhibit more (less) bias than “larger” objects. In this sense, linearity may be thought of as the change in bias over the operational range of the measurement system. In assessing bias, we seek an estimate for the constant difference between the true or reference value and the actual measurement average.

Reproducibility is a factor that affects variation in the mean response of individual groups of measurements. The groups are often distinguished by appraiser (who operates the system), facility (where the measurements are made), or system (what measurement system was used). Other factors used to distinguish groups may be used. Here again, the user

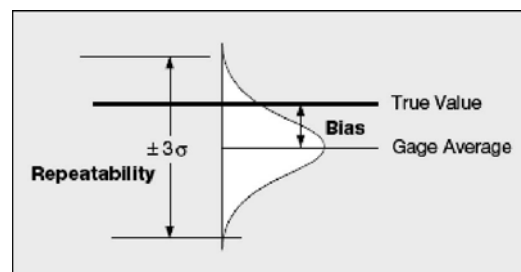


FIG. 1—Repeatability and bias concepts.

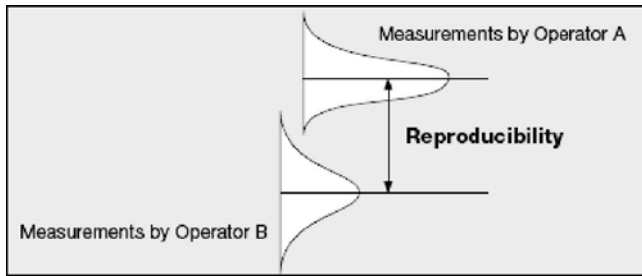


FIG. 2—Reproducibility concept.

of the system must decide what constitutes “reproducibility conditions” for the application being studied. Reproducibility is like a “personal” bias applied equally to every measurement made by the “group.” Each group has its own reproducibility factor that comes from a population of all such “groups” that can be thought to exist. In assessing reproducibility, we seek an estimate of the standard deviation, θ , of this type of random error.

The interpretation of reproducibility may vary in different quarters. In traditional manufacturing, it is the random variation among appraisers (people); in an intralaboratory study, it is the random variation among laboratories. Figure 2 illustrates this concept with “operators” playing the role of the factor of reproducibility.

Stability is variation in bias with time, usually a drift or trend, or erratic type behavior. *Consistency* is a change in repeatability with time. A system is consistent with time when the error due to repeatability remains constant (e.g., is stable). Taken collectively, when a measurement system is stable and consistent, we say that it is a state of statistical control. This further means that we can predict the error of a given measurement within limits.

The best way to study and assess these two properties is to use a control chart technique for averages and ranges. Usually, a number of objects are selected and measured periodically. Each batch of measurements constitutes a subgroup. Subgroups should contain repeated measurements of the same group of objects every time measurements are made in order to capture the variation due to repeatability. Often subgroups are created from a single object measured several times for each subgroup. When this is done, the range control chart will indicate if an inconsistent process is occurring. The average control chart will indicate if the mean is tending to drift or change erratically (stability). Methods discussed in this manual in the section on control charts may be used to judge whether the system is inconsistent or unstable. Figure 3 illustrates the stability concept.

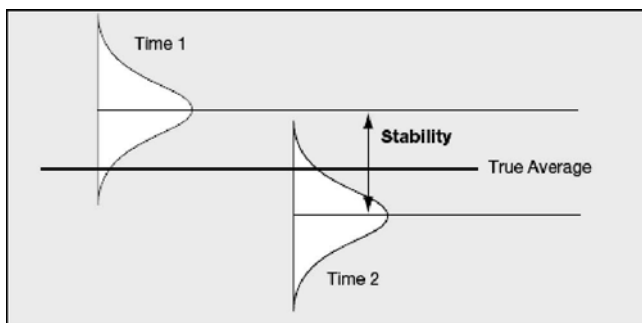


FIG. 3—Stability concept.

The *resolution* of a measurement system has to do with its ability to discriminate between different objects. A highly resolved system is one that is sensitive to small changes from object to object. Inadequate resolution may result in identical measurements when the same object is measured several times under identical conditions. In this scenario, the measurement device is not capable of picking up variation due to repeatability (under the conditions defined). Poor resolution may also result in identical measurements when differing objects are measured. In this scenario, the objects themselves may be too close in true magnitude for the system to distinguish between.

For example, one cannot discriminate time in hours using an ordinary calendar since the latter’s smallest degree of resolution is one day. A ruler graduated in inches will be insufficient to discriminate lengths that differ by less than 1 in. The smallest unit of measure that a system is capable of discriminating is referred to as its finite resolution property. A common rule of thumb for resolution is as follows: If the acceptable range of an object’s true measure is R and if the resolution property is u , then $R/u = 10$ or more is considered very acceptable to use the system to render a decision on measurements of the object.

If a measurement system is perfect in every way except for its finite resolution property, then the use of the system to measure a single object will result in an error $\pm u/2$ where u is the resolution property for the system. For example, in measuring length with a system graduated in inches (here, $u = 1$ in.), if a particular measurement is 129 in., the result should be reported as $129 \pm 1/2$ in. When a sample of measurements is to be used collectively, as, for example, to estimate the distribution of an object’s magnitude, then the resolution property of the system will add variation to the true standard deviation of the object distribution. The approximate way in which this works can be derived. Table 1 shows the resolution effect when the resolution property is a fraction, $1/k$, of the true 6σ span of the object measured, the true standard deviation is 1, and the distribution is of the normal form.

TABLE 1—Behavior of the Measurement Variance and Standard Deviation for Selected Finite Resolution $1/k$ When the True Process Variance is 1 and the Distribution is Normal

k	Total Variance	Resolution Component	Std Dev Due to Component
2	1.36400	0.36400	0.60332
3	1.18500	0.18500	0.43012
4	1.11897	0.11897	0.34492
5	1.08000	0.08000	0.28284
6	1.05761	0.05761	0.24002
8	1.04406	0.04406	0.20990
9	1.03549	0.03549	0.18839
10	1.01877	0.01877	0.13700
12	1.00539	0.00539	0.07342
15	1.00447	0.00447	0.06686

For example, if the resolution property is $u = 1$, then $k = 6$ and the resulting total variance would be increased to 1.0576, giving an error variance due to resolution deficiency of 0.0576. The resulting standard deviation of this error component would then be 0.2402. This is 24% of the true object sigma. It is clear that resolution issues can significantly impact measurement variation.

4.3. SIMPLE REPEATABILITY MODEL

The simplest kind of measurement system variation is called *repeatability*. In its simplest form, it is the variation among measurements made on a single object at approximately the same time under the same conditions. We can think of any object as having a “true” value or that value that is most representative of the truth of the magnitude sought. Each time an object is measured, there is added variation due to the factor of repeatability. This may have various causes, such as nuances in the device setup, slight variations in method, temperature changes, etc. For several objects, we can represent this mathematically as:

$$y_{ij} = x_i + \varepsilon_{ij} \quad (1)$$

Here, y_{ij} represents the j th measurement of the i th object. The i th object has a “true” or reference value represented by x_i , and the repeatability error term associated with the j th measurement of the i th object is specified as a random variable, ε_{ij} . We assume that the random error term has some distribution, usually normal, with mean 0 and some unknown repeatability variance σ^2 . If the objects measured can be conceived as coming from a distribution of every such object, then we can further postulate that this distribution has some mean, μ , and variance θ^2 . These quantities would apply to the true magnitude of the objects being measured.

If we can further assume that the error terms are independent of each other and of the x_i , then we can write the variance component formula for this model as:

$$v^2 = \theta^2 + \sigma^2 \quad (2)$$

Here, v^2 is the variance of the population of all such measurements. It is decomposed into variances due to the true magnitudes, θ^2 , and that due to repeatability error, σ^2 . When the objects chosen for the MSA study are a random sample from a population or a process each of the variances discussed above can be estimated; however, it is not necessary, nor even desirable, that the objects chosen for a measurement study be a random sample from the population of all objects. In theory this type of study could be carried out with a single object or with several specially selected objects (not a random sample). In these cases only the repeatability variance may be estimated reliably.

In special cases, the objects for the MSA study may have known reference values. That is, the x_i terms are all known, at least approximately. In the simplest of cases, there are n reference values and n associated measurements. The repeatability variance may be estimated as the average of the squared error terms:

$$q_1 = \frac{\sum_{i=1}^n (y_i - x_i)^2}{n} = \frac{\sum_{i=1}^n e_i^2}{n} \quad (3)$$

If repeated measurements on either all or some of the objects are made, these are simply averaged all together increasing the degrees of freedom to however many measurements we have.

Let n now represent the total of all measurements. Under the conditions specified above, nq_1/σ^2 has a chi-squared distribution with n degrees of freedom, and from this fact a confidence interval for the true repeatability variance may be constructed.

Example 1

Ten bearing races, each of known inner race surface roughness, were measured using a proposed measurement system. Objects were chosen over the possible range of the process that produced the races.

Reference values were determined by an independent metrology lab on the best equipment available for this purpose. The resulting data and subcalculations are shown in Table 2.

Using Eq 3, we calculate the estimate of the repeatability variance: $q_1 = 0.01674$. The estimate of the repeatability standard deviation is the square root of q_1 . This is

$$\hat{\sigma} = \sqrt{q_1} = \sqrt{0.01674} = 0.1294 \quad (4)$$

When reference values are not available or used, we have to make at least two repeated measurements per object. Suppose we have n objects and we make two repeated measurements per object. The repeatability variance is then estimated as:

$$q_2 = \frac{\sum_{i=1}^n (y_{i1} - y_{i2})^2}{2n} \quad (5)$$

TABLE 2—Bearing Race Data—with Reference Standards

x	y	$(y - x)^2$
0.73	0.80	0.0046
0.91	1.10	0.0344
1.85	1.62	0.0534
2.34	2.29	0.0024
3.11	3.11	0.0000
3.77	4.06	0.0838
3.94	3.96	0.0003
5.29	5.42	0.0180
5.88	5.91	0.0007
6.37	6.44	0.0053
9.11	9.05	0.0040
9.83	10.02	0.0348
11.33	11.36	0.0012
11.89	11.94	0.0021
12.12	12.04	0.0060

Under the conditions specified above, nq_2/σ^2 has a chi-squared distribution with n degrees of freedom, and from this fact a confidence interval for the true repeatability variance may be constructed.

Example 2

Suppose for the data of *Example 1*, we did not have the reference standards. In place of the reference standards, we take two independent measurements per sample, making a total of 30 measurements. This data and the associated differences are shown in Table 3.

Using Eq 4, we calculate the estimate of the repeatability variance: $q_1 = 0.01377$. The estimate of the repeatability standard deviation is the square root of q_1 . This is

$$\hat{\sigma} = \sqrt{q_1} = \sqrt{0.01377} = 0.11734 \quad (6)$$

Notice that this result is close to the result obtained using the known standards except we had to use twice the number of measurements. When we have more than two repeats per object or a variable number of repeats per object, we can use the pooled variance of the several measured objects as the estimate of repeatability. For example, if we have n objects and have measured each object m times each, then repeatability is estimated as:

$$q_3 = \frac{\sum_{i=1}^n \sum_{j=1}^m (y_{ij} - \bar{y}_i)^2}{n(m-1)} \quad (7)$$

Here \bar{y}_i represents the average of the m measurements of object i . The quantity $n(m-1)q_3/\sigma^2$ has a chi-squared distribution with $n(m-1)$ degrees of freedom. There are numerous variations on the theme of repeatability. Still, the analyst must decide what the repeatability conditions are for

the given application. The calculated repeatability standard deviation only applies under the accepted conditions of the experiment.

4.4. SIMPLE REPRODUCIBILITY

To understand the factor of reproducibility, consider the following model for the measurement of the i th object by appraiser j at the k th repeat.

$$y_{ijk} = x_i + \alpha_j + \varepsilon_{ijk} \quad (8)$$

The quantity ε_{ijk} continues to play the role of the repeatability error term, which is assumed to have mean 0 and variance σ^2 . Quantity x_i is the true (or reference) value of the object being measured; quantity and α_j is a random reproducibility term associated with "group" j . This last quantity is assumed to come from a distribution having mean 0 and some variance θ^2 . The α_j terms are interpreted as the random "group bias" or offset from the true mean object response. There is, at least theoretically, a universe or population of all possible groups (people, apparatus, systems, laboratories, facilities, etc.) for the application being studied. Each group has its own peculiar offset from the true mean response. When we select a group for the study, we are effectively selecting a random α_j for that group.

The model in Eq. (8) may be set up and analyzed using a classic variance components, analysis of variance technique. When this is done, separate variance components for both repeatability and reproducibility are obtainable. Details for this type of study may be obtained elsewhere [1-4].

4.5. MEASUREMENT SYSTEM BIAS

Reproducibility variance may be viewed as coming from a distribution of the appraiser's personal bias toward measurement. In addition, there may be a global bias present in the MS that is shared equally by all appraisers (systems, facilities, etc.). Bias is the difference between the mean of the overall distribution of all measurements by all appraisers and a "true" or reference average of all objects. Whereas reproducibility refers to a distribution of appraiser averages, bias refers to a difference between the average of a set of measurements and a known or reference value. The measurement distribution may itself be composed of measurements from differing appraisers or it may be a single appraiser that is being evaluated. Thus, it is important to know what conditions are being evaluated.

Measurement system bias may be studied using known reference values that are measured by the "system" a number of times. From these results, confidence intervals are constructed for the difference between the system average and the reference value. Suppose a reference standard, x , is measured n times by the system. Measurements are denoted by y_i . The estimate of bias is the difference: $\hat{B} = x - \bar{y}$. To determine if the true bias (B) is significantly different from zero, a confidence interval for B may be constructed at some confidence level, say 95%. This formulation is:

$$\hat{B} \pm \frac{t_{\alpha/2} S_y}{\sqrt{n}} \quad (9)$$

In Eq 9, $t_{\alpha/2}$ is selected from Student's t distribution with $n-1$ degrees of freedom for confidence level $C = 1 - \alpha$. If the confidence interval includes zero, we have failed to demonstrate a nonzero bias component in the system.

TABLE 3—Bearing Race Data—Two Independent Measurements, without Reference Standards

y_1	y_2	$(y_1 - y_2)^2$
0.80	0.70	0.009686
1.10	0.88	0.047009
1.62	1.88	0.068959
2.29	2.42	0.017872
3.11	3.29	0.035392
4.06	4.00	0.003823
3.96	3.83	0.015353
5.42	5.18	0.058928
5.91	5.87	0.001481
6.44	6.24	0.042956
9.05	9.26	0.046156
10.02	10.13	0.013741
11.36	11.16	0.040714
11.94	12.04	0.010920
12.04	12.05	0.000016

Example 3: Bias

Twenty measurements were made on a known reference standard of magnitude 12.00. These data are arranged in Table 4.

The estimate of the bias is the average of the $(y - x)$ quantities. This is: $\hat{B} = x - \bar{y} = 0.458$. The confidence interval for the unknown bias, B , is constructed using Eq 9. For 95% confidence and 19 degrees of freedom, the value of t is 2.093. The confidence interval estimate of bias is:

$$0.458 \pm \frac{2.093(0.323)}{\sqrt{20}} \quad (10)$$

$$\rightarrow 0.307 \leq B \leq 0.609$$

In this case, there is a nonzero bias component of at least 0.307.

4.6. USING MEASUREMENT ERROR

Measurement error is used in a variety of ways, and often this is application dependent. We specify a few common uses when the error is of the common repeatability type. If the measurement error is known or has been well approximated, this will usually be in the form of a standard deviation, σ , of error. Whenever a single measurement error is presented, a practitioner or decision maker is always allowed to ask the important question: “What is the error

in this measurement?” For single measurements, and assuming that an approximate normal distribution applies in practice, the 2 or 3-sigma rule can be used. That is, given a single measurement made on a system having this measurement error standard deviation, if x is the measurement, the error is of the form $x \pm 2\sigma$ or $x \pm 3\sigma$. This simply means that the true value for the object measured is likely to fall within these intervals about 95 and 99.7% of the time respectively. For example, if the measurement is $x = 12.12$ and the error standard deviation is $\sigma = 0.13$, the true value of the object measured is probably between 11.86 and 12.38 with 95% confidence or 11.73 and 12.51 with 9.7% confidence.

We can make this interval tighter if we average several measurements. When we use, say, n repeat measurements, the average is still estimating the true magnitude of the object measured and the variance of the average reported will be σ^2/n . The standard error of the average so determined will then be σ/\sqrt{n} . Using the former rule gives us intervals of the form:

$$\bar{x} \pm \frac{2\sigma}{\sqrt{n}}, \text{ or } \bar{x} \pm \frac{3\sigma}{\sqrt{n}} \quad (11)$$

These intervals carry 95 and 99.7% confidence, respectively.

Example 4

A series of eight measurements for a characteristic of a certain manufactured component resulted in an average of 126.89. The standard deviation of the measurement error is known to be approximately 0.8. The customer for the component has stated that the characteristic has to be at least of magnitude 126. Is it likely that the average value reflects a true magnitude that meets the requirement?

We construct a 99.7% confidence interval for the true magnitude, μ . This gives:

$$126.89 \pm \frac{3(0.8)}{\sqrt{8}} \rightarrow 126.04 \leq \mu \leq 127.74 \quad (12)$$

Thus, there is high confidence that the true magnitude μ meets the customer requirement.

4.7. DISTINCT PRODUCT CATEGORIES

We have seen that the finite resolution property (u) of an MS places a restriction on the discriminating ability of the MS (see Section 1.2). This property is a function of the hardware and software system components; we shall refer to it as “mechanical” resolution. In addition, the several factors of measurement variation discussed in this section contribute to further restrictions on object discrimination. This aspect of resolution will be referred to as the *effective resolution*.

The effects of mechanical and statistical resolution can be combined as a single measure of discriminating ability. When the true object variance is τ^2 , and the measurement error variance is σ^2 , the following quantity describes the discriminating ability of the MS.

$$D = \sqrt{\frac{2\tau^2}{\sigma^2} + 1} \approx \frac{1.414\tau}{\sigma} \quad (13)$$

The right-hand side of Eq 13 is the approximation formula found in many texts and software packages. The interpretation of the approximation is as follows. Multiply the

TABLE 4—Bias Data

Reference, x	Measurement, y	$y - x$
12.00	12.657	0.657
12.00	12.461	0.461
12.00	12.715	0.715
12.00	12.724	0.724
12.00	12.740	0.740
12.00	12.669	0.669
12.00	12.065	0.065
12.00	12.665	0.665
12.00	12.125	0.125
12.00	12.643	0.643
12.00	11.625	−0.375
12.00	12.412	0.412
12.00	12.702	0.702
12.00	12.333	0.333
12.00	12.912	0.912
12.00	12.727	0.727
12.00	12.387	0.387
12.00	12.405	0.405
12.00	12.009	0.009
12.00	12.174	0.174

top and bottom of the right-hand member of Eq 13 by 6; rearrange and simplify. This gives:

$$D \approx \frac{6(1.414)\tau}{6\sigma} = \frac{6\tau}{4.24\sigma} \quad (14)$$

The denominator quantity, 4.24σ , is the span of an approximate 97% interval for a normal distribution centered on its mean. The numerator is a similar 99.7% (6-sigma) span for a normal distribution. The numerator represents the true object variation, and the denominator, variation due to measurement error (including mechanical resolution). Then D represents the number of nonoverlapping 97% confidence intervals that fit within the true object variation. This is referred to as the number of distinct product categories or effective resolution within the true object variation.

Illustrations

1. $D = 1$ or less indicates a single category. The system distribution of measurement error is about the same size as the object's true distribution.
2. $D = 2$ indicates the MS is only capable of discriminating two categories. This is similar to the categories "small" and "large."
3. $D = 3$ indicates three categories are obtainable, and this is similar to the categories "small," "medium," and "large."
4. $D \geq 5$ is desirable for most applications.

Great care should be taken in calculating and using the ratio D in practice. First, the values of τ and σ are not typically known with certainty and must be estimated from the results of an MS study. These point estimates themselves carry added uncertainty; second, the estimate of τ is based on the objects selected for the study. If the several objects employed for the study were specially selected and were not a random selection, then the estimate of τ will not represent the true distribution of the objects measured biasing the calculation of D .

Theoretical Background

The theoretical basis for the left-hand side of Eq 13 is as follows. Suppose x and y are measurements of the same object. If each is normally distributed, then x and y have a bivariate normal distribution. If the measurement error has variance σ^2 and the true object has variance τ^2 , then it may be shown that the bivariate correlation coefficient for this case is $\rho = \tau^2/(\tau^2 + \sigma^2)$. The expression for D in Eq 13 is the square root of the ratio $(1 + \rho)/(1 - \rho)$. This ratio is related to the bivariate normal density surface, a function $z = f(x, y)$. Such a surface is shown in 4.

When a plane cuts this surface parallel to the x, y plane, an ellipse is formed. Each ellipse has a major and minor axis. The ratio of the major to the minor axis for the ellipse is the expression for D , Eq 13. The mathematical details of this theory have been sketched by Shewhart [5]. Now consider a set of bivariate x and y measurements from this distribution. Plot the x, y pairs on coordinate paper. First plot the data as the pairs (x, y) . In addition, plot the pairs (y, x) on the same graph. The reason for the duplicate plotting is that there is no reason to use either the x or the y data on either axis. This plot will be symmetrically located about the line $y = x$. If r is the sample correlation coefficient, an ellipse may be constructed and centered on the data. Construction of the

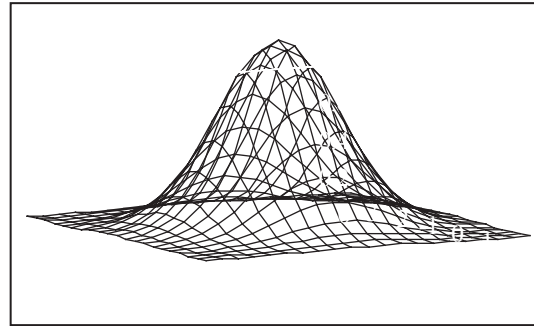


FIG. 4—Typical bivariate normal surface.

ellipse is described by Shewhart [5]. Figure 5 shows such a plot with the ellipse superimposed and the number of distinct product categories shown as squares of side equal to D in Eq 14.

What we see is an elliptical contour at the base of the bivariate normal surface where the ratio of the major to the minor axis is approximately 3. This may be interpreted from a practical point of view in the following way. From 5, the length of the major axis is due principally to the true part variance, while the length of the minor axis is due to repeatability variance alone. To put an approximate length measurement on the major axis, we realize that the major axis is the hypotenuse of an isosceles triangle whose sides we may measure as 6τ (true object variation) each. It follows from simple geometry that the length of the major axis is approximately $1.414(6\tau)$. We can characterize the length of the minor axis simply as 6σ (error variation). The approximate ratio of the major to the minor axis is therefore approximated by discarding the "1" under the radical sign in Eq 13.

PROCESS CAPABILITY AND PERFORMANCE

4.8. INTRODUCTION

Process capability can be defined as the natural or inherent behavior of a stable process. The use of the term "stable

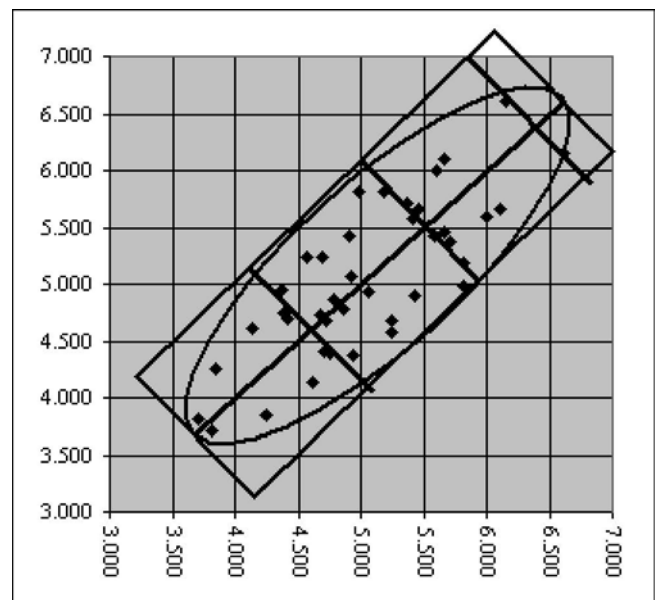


FIG. 5—Bivariate normal surface cross section.

process” may be further thought of as a state of statistical control. This state is achieved when the process exhibits no detectable patterns or trends, such that the variation seen in the data is believed to be random and inherent to the process. This state of statistical control makes prediction possible. Process capability, then, requires process stability or state of statistical control. When a process has achieved a state of statistical control, we say that the process exhibits a stable pattern of variation and is predictable, within limits. In this sense, stability, statistical control, and predictability all mean the same thing when describing the state of a process.

Before evaluation of process capability, a process must be studied and brought under a state of control. The best way to do this is with control charts. There are many types of control charts and ways of using them. **Part 3** of this Manual discusses the common types of control charts in detail. Practitioners are encouraged to consult this material for further details on the use of control charts.

Ultimately, when a process is in a state of statistical control, a minimum level of variation may be reached, which is referred to as common cause or inherent variation. For the purpose of process capability, this variation is a measure of the uniformity of process output, typically a product characteristic.

4.9. PROCESS CAPABILITY

It is common practice to think of process capability in terms of the predicted proportion of the process output falling within product specifications or tolerances. Capability requires a comparison of the process output with a customer requirement (or a specification). This comparison becomes the essence of all process capability measures.

The manner in which these measures are calculated defines the different types of capability indices and their use. For variables data that follow a normal distribution, two process capability indices are defined. These are the “capability” indices and the “performance” indices. Capability and performance indices are often used together but, most important, are used to drive process improvement through continuous improvement efforts. The indices may be used to identify the need for management actions required to reduce common cause variation, to compare products from different sources, and to compare processes. In addition, process capability may also be defined for attribute type data.

It is common practice to define process behavior in terms of its variability. Process capability (PC) is calculated as:

$$PC = 6\sigma_{st} \quad (15)$$

Here, σ_{st} is the standard deviation of the inherent and short-term variability of a controlled process. Control charts are typically used to achieve and verify process control as well as in estimating σ_{st} . The assumption of a normal distribution is not necessary in establishing process control; however, for this discussion, the various capability estimates and their implications for prediction require a normal distribution (a moderate degree of non-normality is tolerable). The estimate of variability over a short time interval (minutes, hours, or a few batches) may be calculated from the within-subgroup variability. This short-term estimate of variation is highly dependent on the manner in which the subgroups were constructed for purposes of the control chart (rational subgroup concept).

The estimate of σ_{st} is:

$$\hat{\sigma}_{st} = \frac{\bar{R}}{d_2} = \frac{\overline{MR}}{d_2} \quad (16)$$

In Eq 16, \bar{R} is the average range from the control chart. When the subgroup size is 1 (individuals chart), the average of the moving range (\overline{MR}) may be substituted. Alternatively, when subgroup standard deviations are used in place of ranges, the estimate is:

$$\hat{\sigma}_{st} = \frac{\bar{s}}{c_4} \quad (17)$$

In Eq 17, \bar{s} is the average of the subgroup standard deviations. Both d_2 and c_4 are a function of the subgroup sample size. Tables of these constants are available in this Manual. Process capability is then computed as:

$$6\hat{\sigma}_{st} = \frac{6\bar{R}}{d_2} \text{ or } \frac{6\overline{MR}}{d_2} \text{ or } \frac{6\bar{s}}{c_4} \quad (18)$$

Let the bilateral specification for a characteristic be defined by the upper (USL) and lower (LSL) specification limits. Let the tolerance for the characteristic be defined as $T = USL - LSL$. The process capability index C_p is defined as:

$$C_p = \frac{\text{specification tolerance}}{\text{process capability}} = \frac{T}{6\hat{\sigma}_{st}} \quad (19)$$

Because the tail area of the distribution beyond specification limits measures the proportion of defective product, a larger value of C_p is better. There is a relation between C_p and the process percent nonconforming only when the process is centered on the tolerance and the distribution is normal. Table 5 shows the relationship.

From Table 5, one can see that any process with a $C_p < 1$ is not as capable of meeting customer requirements (as indicated by percent defectives) compared to a process with $C_p > 1$. Values of C_p progressively greater than 1 indicate more capable processes. The current focus of modern quality is on process improvement with a goal of increasing product uniformity about a target. The implementation of this focus is to create processes having $C_p > 1$. Some industries consider $C_p = 1.33$ (an 8σ specification tolerance) a minimum with a $C_p = 1.66$ (a 10σ specification tolerance) preferred [1]. Improvement of C_p should depend on a company's quality focus, marketing plan, and their competitor's achievements, etc. Note that C_p is also used in process design by design engineers to guide process improvement efforts.

TABLE 5—Relationship among C_p , % Defective and parts per million (ppm) Metric

C_p	%Defective	ppm	C_p	% Defective	ppm
0.6	7.19	71,900	1.10	0.0967	967
0.7	3.57	35,700	1.20	0.0320	318
0.8	1.64	16,400	1.30	0.0096	96
0.9	0.69	6,900	1.33	0.0064	64
1.0	0.27	2,700	1.67	0.0001	0.57

4.10. PROCESS CAPABILITY INDICES ADJUSTED FOR PROCESS SHIFT, C_{pk}

For cases where the process is not centered, the process is deliberately run off-center for economic reasons, or only a single specification limit is involved, C_p is not the appropriate process capability index. For these situations, the C_{pk} index is used. C_{pk} is a process capability index that considers the process average against a single or double-sided specification limit. It measures whether the process is capable of meeting the customer's requirements by considering the specification limit(s), the current process average, and the current short-term process capability σ_{st} . Under the assumption of normality, C_{pk} is estimated as:

$$C_{pk} = \min \left\{ \frac{\bar{x} - LSL}{3\hat{\sigma}_{st}}, \frac{USL - \bar{x}}{3\hat{\sigma}_{st}} \right\} \quad (20)$$

Where a one-sided specification limit is used, we simply use the appropriate term from [6]. The meaning of C_p and C_{pk} is best viewed pictorially as shown in 6.

The relationship between C_p and C_{pk} can be summarized as follows. (a) C_{pk} can be equal to but never larger than C_p ; (b) C_p and C_{pk} are equal only when the process is centered on target; (c) if C_p is larger than C_{pk} , then the process is not centered on target; (d) if both C_p and C_{pk} are >1 , the process is capable and performing within the specifications; (e) if both C_p and C_{pk} are <1 , the process is not capable and not performing within the specifications; and (f) if C_p is >1 and C_{pk} is <1 , the process is capable, but not centered and not performing within the specifications.

By definition, C_{pk} requires a normal distribution with a spread of three standard deviations on either side of the mean. One must keep in mind the theoretical aspects and assumptions underlying the use of process capability indices.

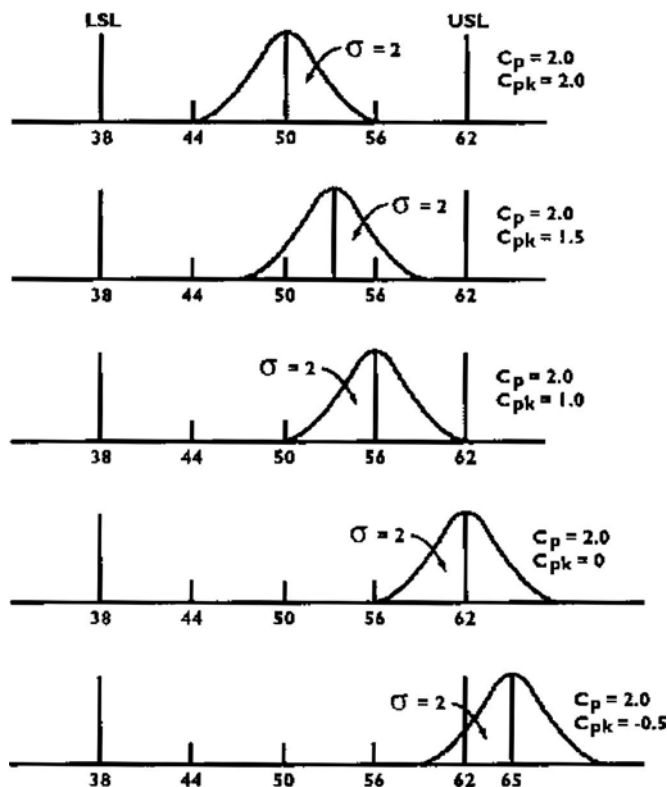


FIG. 6—Relationship between C_p and C_{pk} .

For interpretability, C_{pk} requires a Gaussian (normal or bell-shaped) distribution or one that can be transformed to a normal form. The process must be in a reasonable state of statistical control (stable over time with constant short-term variability). Large sample sizes (preferably greater than 200 or a minimum of 100) are required to estimate C_{pk} with an adequate degree of confidence (at least 95%). Small sample sizes result in considerable uncertainty as to the validity of inferences from these metrics.

4.11. PROCESS PERFORMANCE ANALYSIS

Process performance represents the actual distribution of product and measurement variability over a long period of time, such as weeks or months. In process performance, the actual performance level of the process is estimated rather than its capability when it is in control. As in the case of process capability, it is important to estimate correctly the process variability. For process performance, the long-term variation, σ_{LT} , is developed using accumulated variation from individual production measurement data collected over a long period of time. If measurement data are represented as $x_1, x_2, x_3, \dots, x_n$, the estimate of σ_{LT} is the ordinary sample standard deviation, s , computed from n individual measurements

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}} \quad (21)$$

For a long enough time period, this standard deviation contains the several long-term “components” of variability: (a) lot-to-lot, long-term variability; (b) within-lot, short-term variability; (c) MS variability over the long term; and (d) MS variability over the short term. If the process were in the state of statistical control throughout the period represented by the measurements, one would expect the estimates of short-term and long-term variation to be very close. In a perfect state of statistical control one would expect that the two estimates would be almost identical. According to Ott, Schilling and Neubauer [6] and Gunter [7], this perfect state of control is unrealistic since control charts may not detect small changes in a process. Process performance is defined as $P_p = 6\sigma_{LT}$ where σ_{LT} is estimated from the sample standard deviation S . The performance index P_p is calculated from Eq 22.

$$P_p = \frac{USL - LSL}{6s} \quad (22)$$

The interpretation of P_p is similar to that of C_p . The performance index P_p simply compares the specification tolerance span to process performance. When $P_p \geq 1$, the process is expected to meet the customer specification requirements in the long run. This would be considered an average or marginal performance. A process with $P_p < 1$ cannot meet specifications all the time and would be considered unacceptable. For those cases where the process is not centered, deliberately run off-center for economic reasons, or only a single specification limit is involved, P_{pk} is the appropriate process performance index.

P_p is a process performance index adjusted for location (process average). It measures whether the process is actually meeting the customer's requirements by considering the specification limit(s), the current process average, and the current variability as measured by the long-term standard

deviation (Eq 21). Under the assumption of overall normality, P_{pk} is calculated as

$$P_{pk} = \min \left\{ \frac{\bar{x} - LSL}{3s}, \frac{USL - \bar{x}}{3s} \right\} \quad (23)$$

Here, LSL , USL , and \bar{x} have the same meaning as in the metrics for C_p and C_{pk} . The value of s is calculated from Eq 21. Values of P_{pk} have an interpretation similar to those for C_{pk} . The difference is that P_{pk} represents how the process is running with respect to customer requirements over a specified long time period. One interpretation is that P_{pk} represents what the producer makes and C_{pk} represents what the producer could make if its process were in a state of statistical control. The relationship between P_p and P_{pk} is also similar to that of C_p and C_{pk} .

The assumptions and caveats around process performance indices are similar to those for capability indices. Two obvious differences pertain to the lack of statistical control and the use of long-term variability estimates. Generally, it makes sense to calculate both a C_{pk} and a P_{pk} -like statistic when assessing process capability. If the process is in a state of statistical control then these two metrics will have values

that are very close; alternatively, when C_{pk} and P_{pk} differ in large degree, this indicates that the process was probably not in a state of statistical control at the time the data were obtained.

REFERENCES

- [1] Montgomery, D.C., Borror, C.M. and Burdick, R.K., "A Review of Methods for Measurement Systems Capability Analysis," J. Qual. Technol., Vol. 35. No. 4, 2003.
- [2] Montgomery, D.C., *Design and Analysis of Experiments*, 6th ed., John Wiley & Sons, New York, 2004.
- [3] Automotive Industry Action Group (AIAG), Detroit, MI, FORD Motor Company, General Motors Corporation, and Chrysler Corporation, *Measurement Systems Analysis (MSA) Reference Manual*, 3rd ed., 2003.
- [4] Wheeler, D.J. and Lyday, R.W., *Evaluating the Measurement Process*, SPC Press, Knoxville, TN 2003.
- [5] Shewhart, W.A., *Economic Control of Quality of Manufactured Product* Van Nostrand, New York, 1931, republished by ASQC Quality Press, Milwaukee, WI, 1980.
- [6] Ott, E.R., Schilling, E.G. and Neubauer, D.V., *Process Quality Control*, 4th ed., McGraw-Hill, New York, 2005, pp. 262–268.
- [7] Gunter, B., "The Use and Abuse of C_{pk} ," Qual. Progr., Statistics Corner, January, March, May, and July 1989 and January 1991.

Appendix

List of Some Related Publications on Quality Control

ASTM STANDARDS

- E29-93a (1999), Standard Practice for Using Significant Digits in Test Data to Determine Conformance with Specifications.
 E122-00 (2000), Standard Practice for Calculating Sample Size to Estimate, With a Specified Tolerable Error, the Average for Characteristic of a Lot or Process.

TEXTS

- Bennett, C.A. and Franklin, N.L., *Statistical Analysis in Chemistry and the Chemical Industry*, New York, 1954.
 Bothe, D., *Measuring Process Capability*, McGraw-Hill, New York, 1997.
 Bowker, A.H. and Lieberman, G.L., *Engineering Statistics*, 2nd ed., Prentice-Hall, Englewood Cliffs, NJ 1972.*
 Box, G.E.P., Hunter, W.G. and Hunter, J.S., *Statistics for Experimenters*, Wiley, New York, 1978.
 Burr, I.W., *Statistical Quality Control Methods*, Marcel Dekker, Inc., New York, 1976.*
 Carey, R.G. and Lloyd, R.C., *Measuring Quality Improvement in Healthcare: A Guide to Statistical Process Control Applications*, ASQ Quality Press, Milwaukee, 1995.*
 Cramer, H., *Mathematical Methods of Statistics*, Princeton University Press, Princeton, NJ, 1946.
 Dixon, W.J. and Massey, F.J., Jr., *Introduction to Statistical Analysis*, 4th ed., McGraw-Hill, New York, 1983.
 Duncan, A.J., *Quality Control and Industrial Statistics*, 5th ed., Richard D. Irwin, Inc., Homewood, IL, 1986.*
 Feller, W., *An Introduction to Probability Theory and Its Application*, 3rd ed., Wiley, New York, Vol. 1, 1970, Vol. 2, 1971.
 Grant, E.L. and Leavenworth, R.S., *Statistical Quality Control*, 7th ed., McGraw-Hill, New York, 1996.*
 Guttman, I., Wilks, S.S. and Hunter, J.S., *Introductory Engineering Statistics*, 3rd ed., Wiley, New York, 1982.
 Hald, A., *Statistical Theory and Engineering Applications*, Wiley, New York, 1952.
 Hoel, P.G., *Introduction to Mathematical Statistics*, 5th ed., Wiley, New York, 1984.
 Jenkins, L., *Improving Student Learning: Applying Deming's Quality Principles in Classrooms*, ASQ Quality Press, Milwaukee, 1997.*

- Juran, J.M. and Godfrey, A.B., *Juran's Quality Control Handbook*, 5th ed., McGraw-Hill, New York, 1999.*
 Mood, A.M., Graybill, F.A., and Boes, D.C., *Introduction the Theory of Statistics*, 3rd ed., McGraw-Hill, New York, 1974.
 Moroney, M.J., *Facts from Figures*, 3rd ed., Penguin, Baltimore, MD, 1956.
 Ott, E., Schilling, E.G. and Neubauer, D.V., *Process Quality Control*, 4th ed., McGraw-Hill, New York, 2005.*
 Rickmers, A.D. and Todd, H.N., *Statistics—An Introduction*, McGraw-Hill, New York, 1967.*
 Selden, P.H., *Sales Process Engineering*, ASQ Quality Press, Milwaukee, 1997.
 Shewhart, W.A., *Economic Control of Quality of Manufactured Product*, Van Nostrand, New York, 1931.*
 Shewhart, W.A., *Statistical Method from the Viewpoint of Quality Control*, Graduate School of the U.S. Department of Agriculture, Washington, DC, 1939.*
 Simon, L.E., *An Engineer's Manual of Statistical Methods*, Wiley, New York, 1941.*
 Small, B.B., ed., *Statistical Quality Control Handbook*, AT&T Technologies, Indianapolis, IN, 1984.*
 Snedecor, G.W. and Cochran, W.G., *Statistical Methods*, 8th ed., Iowa State University, Ames, IA, 1989.
 Tippet, L.H.C., *Technological Applications of Statistics*, Wiley, New York, 1950.
 Wadsworth, H.M., Jr., Stephens, K.S. and Godfrey, A.B., *Modern Methods for Quality Control and Improvement*, Wiley, New York, 1986.*
 Wheeler, D.J. and Chambers, D.S., *Understanding Statistical Process Control*, 2nd ed., SPC Press, Knoxville, 1992.*

JOURNALS

- Annals of Statistics*
Applied Statistics (Royal Statistics Society, Series C)
Journal of the American Statistical Association
*Journal of Quality Technology**
Journal of the Royal Statistical Society, Series B
*Quality Engineering**
*Quality Progress**
*Technometrics**

* With special reference to quality control.

INDEX

Index Terms

Links

Note: Page references followed by “*f*” and “*t*” denote figures and tables, respectively.

A

alpha risk	44			
Anderson-Darling (AD) test	23			
appraiser	86			
appraiser variation (AV)	86			
arithmetic mean. <i>See</i> average				
assignable causes	38	40		
attributes, control chart for				
no standard given	46			
standard given	50			
average (<i>X</i>)	14			
vs. average and standard deviation, essential				
information presentation	25			
control chart for, no standard given				
large samples	43	43 <i>t</i>	54	55 <i>f</i>
	56 <i>f</i>	55 <i>t</i>	56 <i>t</i>	
small samples	44	44 <i>t</i>	55	56 <i>t</i>
	57 <i>f</i>	58	58 <i>f</i>	
control chart for, standard given	50	64	64 <i>t</i>	65 <i>f</i>
information in	16			
standard deviation of	77			
uncertainty of. <i>See</i> uncertainty of observed average				
average deviation	15			

B

beta risk	44			
bias	87	87 <i>f</i>	90	91 <i>t</i>
bin				
boundaries	7			
classifying observations into	10 <i>f</i>			
definition of	7			
frequency for	7			

Index Terms

Links

bin (*Cont.*)

number of	7	
rules for constructing	7	9
box-and-whisker plot	12	13 ^f
Box-Cox transformations	24	25

C

capability indices	86	93	
central limit theorem	17		
central tendency, measures of	14		
chance causes	38	40	
Chebyshev's inequality	17 ^t		
coded observations	12		
coefficient of variation (<i>cv</i>)	14		
information in	20	20 ^t	
common causes. <i>See</i> chance causes			
confidence limits	30	31 ^f	31 ^t
use of	32		
consistency	88		
control chart method	38		
breaking up data into rational subgroups	41		
control limits and criteria of control	41		
examples	54		
factors, approximation to	81		
features of	43 ^f		
general technique of	41		
grouping of observations	40 ^t		
for individuals	53		
factors for computing control limits	81		
using moving ranges	54	54 ^t	
using rational subgroups	53	54 ^t	
mathematical relations and tables of factors for	77	78 ^t	81
purpose of	39		
no standard given	43	49 ^t	
for attributes data	46		
for averages and averages and ranges, small			
samples	44		
for averages and standard deviations, large samples	43		

Index Terms

Links

control chart method (*Cont.*)

for averages and standard deviations, small samples	44	44t	
factors for computing control chart lines	45t		
fraction nonconforming	46	47t	
nonconformities per unit	47	48t	
for number of nonconforming units	47	47t	
number of nonconformities	48	48t	49t
risks and	43		
standard given	49	54t	
for attributes data	50		
for averages and standard deviation	50	50t	
factors for computing control chart lines	52t		
fraction nonconforming	50	51t	
nonconformities per unit	52	52t	
for number of nonconforming units	52	52t	
number of nonconformities	52	52t	
for ranges	50	50t	
terminology and technical background	40		
uses of	41		
cumulative frequency distribution	10	11f	
cumulative relative frequency function	12	16	

D

data presentation	1		
application of	2		
data types	2	3t	
essential information	25		
examples	3t	4	
frequency distribution, functions of	13		
graphical presentation	10	11f	
grouped frequency distribution	7		
homogenous data	2	4	
probability plot	21		
recommendations for	1	28	
relevant information	27		
tabular presentation	9t	10	11t
transformations	24		

Index Terms**Links**data presentation (*Cont.*)

ungrouped frequency distribution	4	4f	5t
dispersion, measures of	14		

E

effective resolution	91	
empirical percentiles	6	6f
equipment variation (EV)	86	
essential information	25	27t
definition of	25	
functions that contain	25	
observed relationships	26	26f
presentation of	26t	
expected value	2	

F

fraction nonconforming (p)	14	39		
control chart for				
no standard given	46	47t	59	59f
	59t	60	60f	60t
standard given	50	51t	67	67f
	69f	69t	70t	71f
standard deviation of	80			
frequency bar chart	10			
frequency distribution				
characteristics of	13	13f		
computation of	15	16f		
cumulative frequency distribution	10	11f		
functions of	13			
information in	15			
grouped	7	8t		
ordered stem and leaf diagram	12	13f		
“stem and leaf” diagram	12	12f		
ungrouped	4	4f	5t	
frequency histogram	10			
frequency polygon	10			

Index Terms

Links

G

Gage	86			
gage bias	86			
gage consistency	86			
gage linearity	86			
gage R&R	86			
gage repeatability	86			
gage reproducibility	86			
gage resolution	86			
gage stability	86			
geometric mean	14			
goodness of fit tests	23			
grouped frequency distribution	7	8 <i>t</i>		
cumulative frequency distribution	10	11 <i>f</i>		
definitions of	7			
graphical presentation	10	11 <i>f</i>		
tabular presentation	9 <i>t</i>	10	11 <i>t</i>	

H

homogenous data	2	4		
-----------------	---	---	--	--

I

individual observations				
control chart for	53			
using moving ranges	54	54 <i>t</i>	75	75 <i>f</i>
	76 <i>f</i>			
using rational subgroups	53	54 <i>t</i>	73	73 <i>f</i>
	73 <i>t</i>	75 <i>f</i>		
intermediate precision	87			
interquartile range (IQR)	12			

K

kurtosis (g_2)	13	14 <i>f</i>	15	
information in	18			

Index Terms

Links

L

leptokurtic distribution	15
linearity	87
long-term variability	86
lopsidedness	
measures of	15
lot	38
lower quartile (Q_1)	12

M

measurement, definition of	86	
measurement error	91	
measurement process	87	
measurement system	86	
basic properties	87	
bias	90	91 _t
distinct product categories	91	
measurement error	91	
resolution of	88	88 _t
simple repeatability model	89	89 _t
simple reproducibility model	90	
measurement systems analysis (MSA)	87	
mechanical resolution	91	
median	6	12
mesokurtic distribution	15	
Minitab	24	

N

nonconforming unit	46			
nonconformity	46			
per unit (u)				
control chart for, no standard given	47	48 _t	61	61 _f
	62 _f	61 _t	62 _t	
control chart for, standard given	52	52 _t	71	71 _t
	72 _f			
standard deviation of	81			
normal probability plot	22	22 _f		

Index Terms

Links

number of nonconforming units (np)

control chart for

no standard given

47

47 t

59 f

60

60 t

standard given

52

52 t

67

68 f

69 t

number of nonconformities (c)

control chart for

no standard given

48

48 t

49 t

61

61 t

62 f

62 t

63

63 t

64 f

standard given

52

52 t

72

72 f

72 t

standard deviation of

81

O

ogive

11

one-sided limit

32

ordered stem and leaf diagram

12

13 f

order statistics

6

outliers

12

20

P

peakedness

measures of

15

percentile

6

performance indices

86

93

platykurtic distribution

15

power transformations

24

24 t

probability plot

21

definition of

21

normal distribution

21

22 f

22 t

Weibull distribution

23

23 f

23 t

probable error

29

process capability (C_p)

92

definition of

86

92

indices adjusted for process shift

94

Index Terms

Links

process performance (P_p)	86	94		
process shift (C_{pk})				
and process capability, relationship between	94	94 f		
Q				
quality characteristics	2			
R				
range (R)	15			
control chart for, no standard given				
small samples	44	44 t	58	58 f
control chart for, standard given	50	67	67 f	67 t
standard deviation of	80			
rank regression	23			
reference value	87			
relative error	15			
relative frequency (p)	14			
single percentile of	16	16 f		
values of	16			
relative standard deviation	15	20		
relevant information	27			
evidence of control	27			
repeatability	87	87 f	89	89 t
reproducibility	87	88 f	91	
root-mean-square deviation ($s_{(rms)}$)	14			
rounding-off procedure	33	34	34 f	
S				
“s” graph	11			
sample, definition of	38			
Shewhart, Walter	42			
short-term variability	86			
skewness (g_1)	13	13 f	15	
information in	18			
special causes. <i>See</i> assignable causes				
stability	88	88 f		
stable process	92			

Index Terms

Links

standard deviation (s)	14			
control chart for, no standard given				
large samples	43	43 _t	54	55 _f
	56 _f	55 _t	56 _t	
small samples	44	44 _t	55	56 _t
	57 _f			
control chart for, standard given	50	64	64 _t	65 _f
for control limits, basis of	80 _t			
information in	17			
standard deviation of	77	80		
statistical control	27	86		
lack of	40			
statistical probability	30			
“stem and leaf” diagram	12	12 _f		
Stirling’s formula				
Sturge’s rule,				
subgroup, definition of	38	39		
sublot	39			

T

3-sigma control limits	41	
tolerance limits	20	
transformations	24	
Box-Cox transformations	24	25
power transformations	24	24 _t
use of	25	
true value	87	

U

uncertainty of observed average		
computation of limits	30	31 _t
data presentation	31	32 _f
experimental illustration	30	32 _f
for normal distribution (σ)	34	35 _t
number of places of figures	33	
one-sided limits	32	
and systematic/constant error	33 _f	

Index Terms

Links

uncertainty of observed average (<i>Cont.</i>)			
plus or minus limits of	29		
theoretical background	29		
for population fraction	36	36f	
ungrouped frequency distribution	4	4f	5t
empirical percentiles and order statistics	6	6f	
unit	39		
upper quartile (Q_3)	12		

V

variance	14		
reproducibility	90		
variance-stabilizing transformations. <i>See</i> power transformations			

W

warning limits	42		
Weibull probability plot	23	23f	23t
whiskers	12		